

## Geometric Series

Recall:

A **geometric sequence** is a sequence where the ratio of any term and the preceding term is a constant value. This constant value is called the **common ratio** and is usually denoted  $r$ . In general, for any geometric sequence,

$$a_1, a_1r, a_1r^2, a_1r^3, \dots$$

the value of  $r$  is given by

$$r = \frac{a_n}{a_{n-1}}$$

and the  $n$ th term of the geometric sequence is explicitly given by

$$a_n = a_1r^{n-1}$$

As you might expect, a **geometric series** is the sum of the terms of a geometric sequence. Let's find a formula for the sum of the first  $n$  terms of a geometric series.

### Example

Consider the series  $9 - 3 + 1 - \frac{1}{3} + \dots$

Is this a geometric series?

Give an explicit formula for  $a_n$ .

Find the exact value of the sum of the first 16 terms of this series.

Find the approximate value of the first 10 terms, 15 terms, 20 terms, 30 terms and 40 terms of this series. What seems to be the limit of the series?

### Example

For each of the following geometric series (i) write the first seven terms of the series, (ii) find an explicit formula for the  $n$ th term of the series, and (iii) find the exact value of the sum of the first 20 terms of the series.

1.  $1 + 3 + 9 + 27 + \dots$

2.  $\sum_{k=1}^n 7 \left(\frac{1}{2}\right)^k$

3.  $0.9 + 0.09 + 0.009 + 0.0009 + \dots$

4.  $-0.9 + 0.09 - 0.009 + 0.0009 - \dots$

## Infinite Geometric Series

Let's look more closely at the series

$$0.9 + 0.09 + 0.009 + 0.0009 + \dots$$

Here,  $a_1 = \frac{9}{10}$  and  $r = \frac{1}{10}$ . Thus the  $n$ th partial sum of the series (the sum of the first  $n$  terms) is given by

$$S_n = \frac{\frac{9}{10} \left(1 - \left(\frac{1}{10}\right)^n\right)}{1 - \frac{1}{10}}$$

Now, as  $n$  gets larger and larger,  $\left(\frac{1}{10}\right)^n$  gets closer and closer to zero. We say that as  $n \rightarrow \infty$ ,

$$S_n \rightarrow \frac{\frac{9}{10} (1 - 0)^n}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

If  $|r| < 1$ , then the infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \dots$$

has the sum

$$S = \frac{a_1}{1 - r}$$

### Example

Find the sum of the following series. If the sum does not exist, explain why it does not.

1.  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \dots$

2.  $\sum_{k=1}^{\infty} 7 \left(\frac{1}{2}\right)^k$

3.  $\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} \dots$

4.  $-0.9 + 0.09 - 0.009 + 0.0009 - \dots$

**Problems** (Most from Stewart, James *College Algebra* 4e, p. 646)

Find the exact value of the sum of the geometric series.

1.  $1 + 3 + 9 + \cdots + 2187$

2.  $\sum_{k=1}^{11} 3 \left(\frac{1}{2}\right)^{k-1}$

3.  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

4.  $\sum_{k=1}^5 7 \left(\frac{3}{2}\right)^{k-1}$

Find the exact value of the sum of the infinite geometric series.

5.  $1 - \frac{2}{5} + \frac{4}{5} - \frac{8}{125} + \cdots$

6.  $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$

7.  $1 + \frac{1}{3^{1/2}} + \frac{1}{3} + \frac{1}{3^{3/2}} + \cdots$

8.  $a + ab^2 + ab^4 + ab^6 + \cdots$

Evaluate each series, if the sum exists.

9.  $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1}$

11.  $\sum_{k=1}^{\infty} \left(\frac{1}{2} + 2k\right)$

10.  $\sum_{j=0}^{\infty} 12 \left(\frac{9}{11}\right)^j$

12.  $\sum_{j=1}^{\infty} (-1)^j (0.001)^k$

13. Initially, a pendulum swings through an arc with length 2 feet. On each successive swing, the length of the arc is 0.95 of the previous length.

- (a) What is the length of the arc after 20 swings?
- (b) On which swing is the length of the arc first less than 1 foot?
- (c) After 15 swings, what total length will the pendulum have swung?
- (d) When it stops, what total length will the pendulum have swung?