Solve *The system does not have a unique solution. Solve the system if possible.* [1]

$$\begin{cases} x+5y-2z = 5 \\ x+3y-2z = 23 \\ x+4y-2z = 14 \end{cases}$$

Begin by labeling the equations so we can keep track of what we're doing.

$$\begin{cases} x+5y-2z = 5 & \text{(A)} \\ x+3y-2z = 23 & \text{(B)} \\ x+4y-2z = 14 & \text{(C)} \end{cases}$$

The x in equation (A) already has a coefficient of 1 so we can use it to eliminate the x's in equations (B) and (C) as follows:

(B)−(A) → (B) and (C)−(A) → (C)

$$\begin{cases} x+5y-2z = 5 & \text{(A)} \\ \quad -2y \qquad = 18 & \text{(B)} \\ \quad -1y \qquad = 9 & \text{(C)} \end{cases}$$

Multiply (B) by $-1/2$ to get 1 as the coefficient on the y.

$-\frac{1}{2}$ (B) → (B)

$$\begin{cases} x+5y-2z = 5 & \text{(A)} \\ \qquad y \qquad = -9 & \text{(B)} \\ \quad -1y \qquad = 9 & \text{(C)} \end{cases}$$

Add equations (B) and (C) to eliminate y in equation (C).

(C)+(B) → (C)

$$\begin{cases} x+5y-2z = 5 & \text{(A)} \\ \qquad y \qquad = -9 & \text{(B)} \\ \qquad\qquad 0 = 0 & \text{(C)} \end{cases}$$

Notice how equation (C) is the true statement, $0 = 0$. This means that the system has infinitely many solutions.

Since we don't have a solution for z we say that z can be any number and back-substitute from there.

From equation (B) we know that $y = -9$ and we can use this information to solve for x in terms of z using equation (A).

$$\begin{aligned} x+5y-2z &= 5 \\ x+5(-9)-2z &= 5 \\ x &= 5+45+2z \\ x &= 50+2z \\ x &= 2z+50 \end{aligned}$$

There are infinitely many solutions of the form $(2z+50, -9, z)$.

[1] Harshbarger/Yocco, *College Algebra In Context, 5e*, p. 519 #22.

Another way we can achieve the same result is by using matrices. Begin with the augmented matrix...

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 5 \\ 1 & 3 & -2 & 23 \\ 1 & 4 & -2 & 14 \end{array}\right]$$

and use reduced row-echelon form (rref) on your calculator to get the solution matrix...

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The bottom row of all zeros tells us that the system has infinitely many solutions and we can let z equal any number.

The middle row says $y = -9$ and when we substitute and solve the top row for x we get $x = 2z + 50$.

So there are infinitely many solutions of the form $(2z + 50, \, -9, z)$.