For the given functions f and g, find (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$. State the domain of each composite function.[1]

$$f(x) = \frac{1}{x+3};\ g(x) = -\frac{2}{x}$$

a. $f \circ g$

Let's examine the individual functions first. For $f(x) = \frac{1}{x+3}$, we must exclude $x = -3$ from the domain. For $g(x) = -\frac{2}{x}$, we must exclude $x = 0$ from the domain. Since $g(x)$ is the input to f, we must also be sure that the output from $g(x)$ is never -3, because it becomes the input to $f(x)$. We solve

$$\begin{aligned} g(x) &= -3 \\ -\frac{2}{x} &= -3 \\ -\frac{2}{x} \cdot x &= -3 \cdot x \quad x \neq 0 \\ -2 &= -3x \\ \frac{2}{3} &= x \end{aligned}$$

So in addition to $x \neq 0$, we also know $x \neq \frac{2}{3}$. Again, $x \neq 0$ because the input function $g(x)$ would be undefined and $x \neq \frac{2}{3}$ because with that input to g, the function $f(x)$ would be undefined.

Thus the domain of $f(g(x))$ is all real numbers x such that $x \neq 0$ or $x \neq \frac{2}{3}$. In set builder notation, we write $\left\{x | x \neq 0, x \neq \frac{2}{3}\right\}$. Using interval notation, we write $(-\infty,0) \cup \left(0,\frac{2}{3}\right) \cup \left(\frac{2}{3},\infty\right)$.

Now that we know the domain, lets find the function rule for $f \circ g$.

$$\begin{aligned} f \circ g &= f(g(x)) \\ &= f\left(-\frac{2}{x}\right) \\ &= \frac{1}{\frac{-2}{x} + 3} \\ &= \frac{1}{\frac{-2}{x} + 3 \cdot \frac{x}{x}} \\ &= \frac{1}{\frac{-2+3x}{x}} \\ &= \frac{x}{3x-2} \end{aligned}$$

Thus, if $f(x) = \frac{1}{x+3}$ and $g(x) = -\frac{2}{x}$, then

$$f \circ g = f(g(x)) = \frac{x}{3x-2}$$

and the domain of $f \circ g$ is $(-\infty,0) \cup \left(0,\frac{2}{3}\right) \cup \left(\frac{2}{3},\infty\right)$.

[1]Sullivan, *Precalculus: Enhanced with Graphing Utilities*, p. 257, #36.

b. $g \circ f$

$g \circ f(x) = g(f(x))$. Here, the input is $f(x)$ and from part (**a**) we know that we must exclude $x = -3$ from the domain. For $g(x) = -\frac{2}{x}$, we must exclude $x = 0$ from the domain. Since $f(x)$ is the input to g, we must also be sure that the output from $f(x)$ is never 0, because it becomes the input to $g(x)$. We solve

$$\begin{aligned} f(x) &= 0 \\ \frac{1}{x+3} &= 0 \\ \frac{1}{x+3} \cdot (x+3) &= 0 \cdot (x+3) \quad x \neq -3 \\ 1 &= 0 \end{aligned}$$

This last equation is never true, and that means there are *no* values of x for which $f(x) = 0$.

Thus the domain of $g(f(x))$ is all real numbers x such that $x \neq -3$. In set builder notation, we write $\{x | x \neq -3\}$. Using interval notation, we write $(-\infty, -3) \cup (-3, \infty)$.

Now that we know the domain, lets find the function rule for $g \circ f$.

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g\left(\frac{1}{x+3}\right) \\ &= -\frac{2}{\frac{1}{x+3}} \\ &= -2 \cdot \frac{x+3}{1} \\ &= -2\,(x+3) \end{aligned}$$

Thus, if $f(x) = \frac{1}{x+3}$ and $g(x) = -\frac{2}{x}$, then

$$g \circ f = g(f(x)) = -2\,(x+3)$$

and the domain of $g \circ f$ is $(-\infty, -3) \cup (-3, \infty)$.

c. $f \circ f$

$f \circ f(x) = f(f(x))$. Here, the input is $f(x)$ and from part (**a**) we know that we must exclude $x = -3$ from the domain. Since $f(x)$ is the input to f, we must also be sure that the output from $f(x)$ is never -3, because it becomes the input to $f(x)$. We solve

$$\begin{aligned} f(x) &= -3 \\ \frac{1}{x+3} &= -3 \\ \frac{1}{x+3} \cdot (x+3) &= -3 \cdot (x+3) \qquad x \neq -3 \\ 1 &= -3x - 9 \\ 10 &= -3x \\ -\frac{10}{3} &= x \end{aligned}$$

So in addition to $x \neq -3$, we also know $x \neq -\frac{10}{3}$. Again, $x \neq -3$ because the input function $f(x)$ would be undefined and $x \neq -\frac{10}{3}$ because with that input to f, the function $f(x)$ would be undefined.

Thus the domain of $f(f(x))$ is all real numbers x such that $x \neq -3$ or $x \neq -\frac{10}{3}$. In set builder notation, we write $\left\{x | x \neq -3, x \neq -\frac{10}{3}\right\}$. Using interval notation, we write $\left(-\infty, -\frac{10}{3}\right) \cup \left(-\frac{10}{3}, -3\right) \cup (-3, \infty)$.

Now that we know the domain, lets find the function rule for $f \circ f$.

$$\begin{aligned} f \circ f &= f(f(x)) \\ &= f\left(\frac{1}{x+3}\right) \\ &= \frac{1}{\frac{1}{x+3} + 3} \\ &= \frac{1}{\frac{1}{x+3} + 3 \cdot \frac{x+3}{x+3}} \\ &= \frac{1}{\frac{1+3(x+3)}{x+3}} \\ &= \frac{1}{\frac{3x+10}{x+3}} \\ &= \frac{x+3}{3x+10} \end{aligned}$$

Thus, if $f(x) = \frac{1}{x+3}$, then

$$f \circ f = f(f(x)) = \frac{x+3}{3x+10}$$

and the domain of $f \circ f$ is $\left(-\infty, -\frac{10}{3}\right) \cup \left(-\frac{10}{3}, -3\right) \cup (-3, \infty)$.

d. $g \circ g$

$g \circ g(x) = g(g(x))$. Here, the input is $g(x)$ and from part (**a**) we know that we must exclude $x = 0$ from the domain. Since $g(x)$ is the input to g, we must also be sure that the output from $g(x)$ is never 0, because it becomes the input to $g(x)$. We solve

$$\begin{aligned} g(x) &= 0 \\ -\frac{2}{x} &= 0 \\ -\frac{2}{x} \cdot x &= 0 \cdot x \quad x \neq 0 \\ -2 &= 0 \end{aligned}$$

This last equation is never true, and that means there are *no* values of x for which $g(x) = 0$.

Thus the domain of $g(g(x))$ is all real numbers x such that $x \neq 0$. In set builder notation, we write $\{x|x \neq 0\}$. Using interval notation, we write $(-\infty,0) \cup (0,\infty)$.

Now that we know the domain, lets find the function rule for $g \circ g$.

$$\begin{aligned} g \circ g &= g(g(x)) \\ &= g\left(-\frac{2}{x}\right) \\ &= -\frac{2}{\frac{-2}{x}} \\ &= -2 \cdot \frac{x}{-2} \\ &= \frac{-2x}{-2} \\ &= x \end{aligned}$$

Thus, if $g(x) = -\frac{2}{x}$, then

$$g \circ g = g(g(x)) = x$$

and the domain of $g \circ g$ is $(-\infty,0) \cup (0,\infty)$.