Use the information given about the angle θ, $0 \le \theta < 2\pi$, to find the exact value of the following trigonometric functions.[1]

$$\csc(\theta) = -\sqrt{5} \quad \cos(\theta) < 0$$

We are given $\csc(\theta) = -\sqrt{5}$. So, in terms of more familiar trig functions, we have

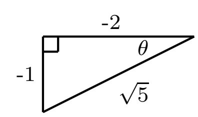
$$\begin{aligned} \csc(\theta) &= -\sqrt{5} \\ \frac{1}{\sin(\theta)} &= -\sqrt{5} \\ 1 &= -\sqrt{5} \cdot \sin(\theta) \end{aligned}$$

or

$$\sin(\theta) = -\frac{1}{\sqrt{5}}$$

This tells us that θ must be in quadrant III or IV. We are also given $\cos(\theta) < 0$, so θ must be in quadrant II or III. Thus we know that θ is in quadrant III.

We use this to draw a triangle to help us find the other trig functions of θ.



From the diagram, we get

$$\begin{aligned} \cos(\theta) &= -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5} \\ \sin(\theta) &= -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5} \\ \tan(\theta) &= \frac{1}{2} \end{aligned}$$

(a) $\sin(2\theta)$

$$\begin{aligned} \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ &= 2 \cdot -\frac{\sqrt{5}}{5} \cdot -\frac{2\sqrt{5}}{5} \\ &= \frac{4}{5} \end{aligned}$$

(b) $\cos(2\theta)$

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \left(-\frac{2\sqrt{5}}{5}\right)^2 - \left(-\frac{\sqrt{5}}{5}\right)^2 \\ &= \frac{4 \cdot 5}{25} - \frac{5}{25} \\ &= \frac{15}{25} \\ &= \frac{3}{5} \end{aligned}$$

[1] Sullivan, *Precalculus: Enhanced with Graphing Utilities*, p. 495, #14.

(c) $\sin\left(\frac{\theta}{2}\right)$

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos(\theta)}{2}} \\
&= \pm\sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{5}{5}+\frac{2\sqrt{5}}{5}}{2}} \\
&= \pm\sqrt{\frac{5+2\sqrt{5}}{5}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{5+2\sqrt{5}}{10}}
\end{aligned}$$

Since we know θ is in quadrant III, we can deduce that $\frac{\theta}{2}$ is in quadrant II. (Since $180^\circ < \theta < 270^\circ$, it must be that $\frac{180^\circ}{2} < \frac{\theta}{2} < \frac{270^\circ}{2}$ or $90^\circ < \frac{\theta}{2} < 135^\circ$. Finally, the sine function is positive in quadrant II, and thus

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{5+2\sqrt{5}}{10}}$$

(d) $\cos\left(\frac{\theta}{2}\right)$

$$\begin{aligned}
\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1+\cos(\theta)}{2}} \\
&= \pm\sqrt{\frac{1+\left(-\frac{2\sqrt{5}}{5}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{5}{5}-\frac{2\sqrt{5}}{5}}{2}} \\
&= \pm\sqrt{\frac{5-2\sqrt{5}}{5}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{5-2\sqrt{5}}{10}}
\end{aligned}$$

From (c) we know $\frac{\theta}{2}$ is in quadrant II, and so $\cos\left(\frac{\theta}{2}\right)$ must be negative. Thus,

$$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5-2\sqrt{5}}{10}}$$

(e) $\tan(2\theta)$

$$\begin{aligned}\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)} \\ &= \frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} \\ &= \frac{1}{1-\frac{1}{4}} \\ &= \frac{4}{3}\end{aligned}$$

(f) $\tan\left(\frac{\theta}{2}\right)$

$$\begin{aligned}\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ &= \pm\sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{1+\left(-\frac{2\sqrt{5}}{5}\right)}} \\ &= \pm\sqrt{\frac{\frac{5}{5}+\frac{2\sqrt{5}}{5}}{\frac{5}{5}-\frac{2\sqrt{5}}{5}}} \\ &= \pm\sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{\frac{5-2\sqrt{5}}{5}}} \\ &= \pm\sqrt{\frac{5+2\sqrt{5}}{5}\cdot\frac{5}{5-2\sqrt{5}}} \\ &= \pm\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}\end{aligned}$$

From (c) we know $\frac{\theta}{2}$ is in quadrant II, and so $\tan\left(\frac{\theta}{2}\right)$ must be negative. Thus,

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}$$

Or

$$\begin{aligned}\tan\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}} \\ &= -\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}\cdot\frac{5+2\sqrt{5}}{5+2\sqrt{5}}} \\ &= -\sqrt{\frac{25+20\sqrt{5}+20}{25-20}} \\ &= -\sqrt{\frac{5\left(9+4\sqrt{5}\right)}{5}} \\ &= -\sqrt{9+4\sqrt{5}}\end{aligned}$$