Calculus I, Section 2.3, #32
Calculating Limits Using the Limit Laws

Evaluate the limit, if it exists.[1]

$$\lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

If we try direct substitution, we get expressions that are undefined. Let's do the algebra to rewrite the difference as a single fraction.

$$\begin{aligned}
\lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} &= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2}\cdot\dfrac{x^2}{x^2} - \dfrac{1}{x^2}\cdot\dfrac{(x+h)^2}{(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x^2}{x^2(x+h)^2} - \dfrac{(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x^2-\left(x^2+2xh+h^2\right)}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{-2xh-h^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{h(-2x-h)}{x^2(x+h)^2}\cdot\frac{1}{h} \\
&= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2}
\end{aligned}$$

and now we do direct substitution

$$\begin{aligned}
&= \frac{-2x-0}{x^2(x+0)^2} \\
&= \frac{-2x}{x^4} \\
&= \frac{-2}{x^3} \\
&= -\frac{2}{x^3}
\end{aligned}$$

Thus,

$$\lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = -\frac{2}{x^3}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 103, #32.