(a) *If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.*[1]

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} \\ &= \lim_{h\to 0} \frac{g(h)-g(0)}{h} \\ &= \lim_{h\to 0} \frac{h^{2/3}-0^{2/3}}{h} \\ &= \lim_{h\to 0} \frac{h^{2/3}}{h} \\ &= \lim_{h\to 0} \frac{1}{h^{1/3}} \end{aligned}$$

Now, if $h \to 0^-$, then $h^{1/3}$ is negative-valued and $\frac{1}{h^{1/3}} \to -\infty$. If $h \to 0^+$, then $h^{1/3}$ is positive-valued and $\frac{1}{h^{1/3}} \to \infty$. Thus the limit as $h \to 0$ does not exist.

(Note that either of these one-sided limits becoming infinite is enough to conclude $g'(0)$ does not exist. More about this in a future lesson.)

(b) *If $a \neq 0$, find $g'(a)$.*

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^{2/3}-a^{2/3}}{h} \end{aligned}$$

and we're stuck. No more algebra is possible to simplify this. Let's try the other definition of derivative.

$$\begin{aligned} g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} \frac{x^{2/3}-a^{2/3}}{x-a} \\ &= \lim_{x\to a} \frac{\left(x^{1/3}\right)^2-\left(a^{1/3}\right)^2}{x-a} \end{aligned}$$

From difference of squares, we get

$$= \lim_{x\to a} \frac{\left(x^{1/3}+a^{1/3}\right)\left(x^{1/3}-a^{1/3}\right)}{x-a}$$

Now we'll write the denominator as a difference of cubes, to look for a common factor with the numerator.

$$\begin{aligned} &= \lim_{x\to a} \frac{\left(x^{1/3}+a^{1/3}\right)\left(x^{1/3}-a^{1/3}\right)}{\left(x^{1/3}\right)^3-\left(a^{1/3}\right)^3} \\ &= \lim_{x\to a} \frac{\left(x^{1/3}+a^{1/3}\right)\left(x^{1/3}-a^{1/3}\right)}{\left(x^{1/3}-a^{1/3}\right)\left(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}\right)} \\ &= \lim_{x\to a} \frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} \\ &= \frac{2a^{1/3}}{3a^{2/3}} \\ &= \frac{2}{3a^{1/3}} \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 164, #58.

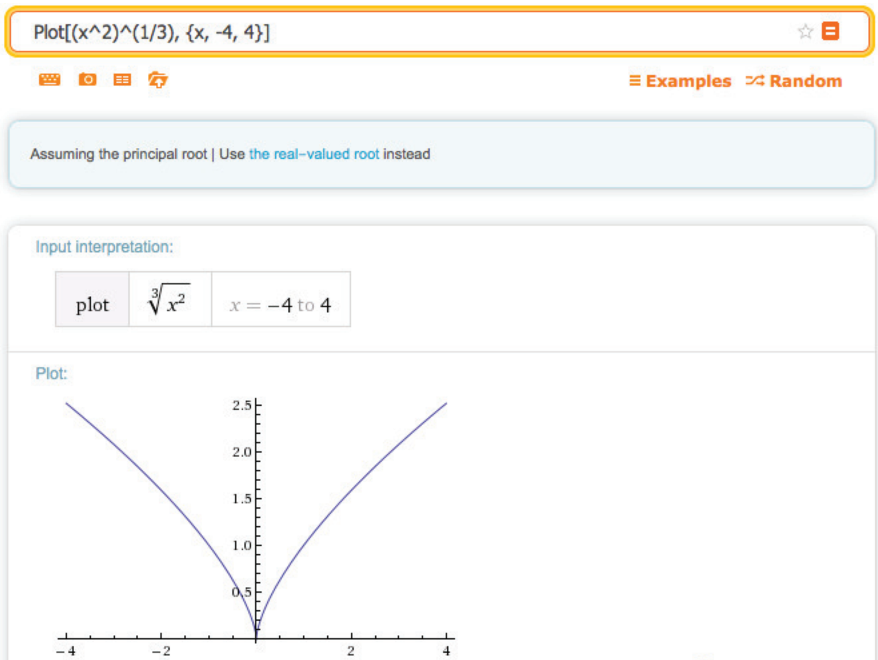
(c) *Show that* $y = x^{2/3}$ *has a vertical tangent line at* (0,0).

Our work in part (a) with the one-sided limits shows that when $a < 0$, the slope of the tangent line is negative and the tangent line becomes steeper with negative slope as we get closer to 0. Similarly, when $a > 0$, the slope of the tangent line is positive and becomes the tangent line becomes steeper with positive slope the close we get to 0.

Thus, there is a vertical tangent line at $x = 0$.

(d) *Illustrate part (c) by by graphing* $y = x^{2/3}$.

Using WolframAlpha, we get



which cleary shows the vertical tangent line at (0,0).