An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2\cos(t) + 3\sin(t)$, $t > 0$, where s is measured in centimeters and t is seconds. (Take the positive direction to be downward.)[1]

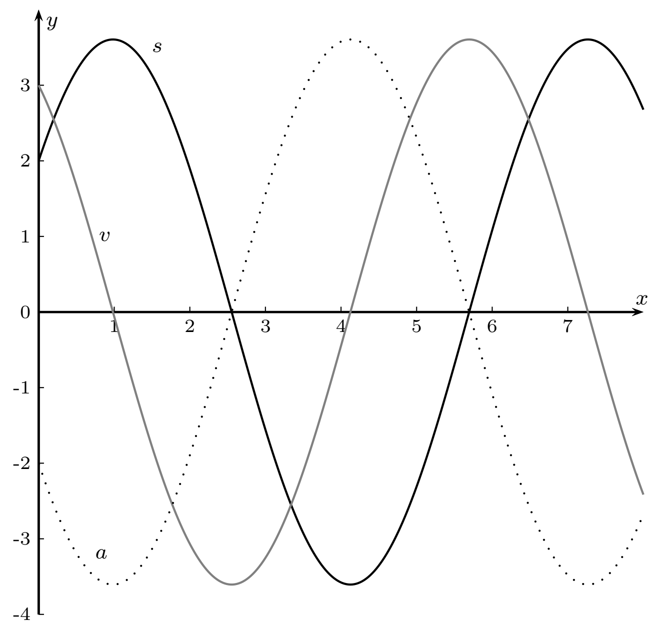
(a) *Find the velocity and acceleration at time t.*

$$s(t) = 2\cos(t) + 3\sin(t)$$

so

$$\begin{aligned} v(t) &= s'(t) = -2\sin(t) + 3\cos(t) \\ a(t) &= v'(t) = -2\cos(t) - 3\sin(t) \end{aligned}$$

(b) *Graph the velocity and acceleration functions.*



(c) *When does the mass pass through the equilibrium position for the first time?*

The "equilibrium position" would be where the mass hangs before it is pulled downward, *i.e.*, at $s = 0$.

From the graph, when $s = 0$, it seems $t \approx 2.5$. For a better result, we solve

$$\begin{aligned} 0 &= 2\cos(t) + 3\sin(t) \\ -2\cos(t) &= 3\sin(t) \\ -2 &= \frac{3\sin(t)}{\cos(t)} \\ -\frac{2}{3} &= \tan(t) \\ \tan^{-1}\left(-\frac{2}{3}\right) &= \tan^{-1}(\tan(t)) \\ -0.5880 &\approx t \end{aligned}$$

Using the inverse tangent has given us a negative value, but we know $t > 0$. The first positive solution to $\tan(t) = -\frac{2}{3}$ will be $\approx -0.5880 + \pi \approx 2.5536$.

[1]Stewart, *Calculus, Early Transcendentals*, p. 196, #36.

(d) *How far from its equilibrium position does the mass travel?*

The mass will reach its maximum distance from zero when its $v = 0$. (The mass is rising upward, reaches a maximum where $v = 0$, and the falls back down.)

From the graph, when $v = 0$, it seems $t \approx 1$. For a better result, we solve

$$\begin{aligned} 0 &= -2\sin(t) + 3\cos(t) \\ -3\cos(t) &= -2\sin(t) \\ -3 &= \frac{-2\sin(t)}{\cos(t)} \\ \frac{3}{2} &= \tan(t) \\ \tan^{-1}\left(\frac{3}{2}\right) &= \tan^{-1}(\tan(t)) \\ 0.9828 &\approx t \end{aligned}$$

Using the inverse tangent has given us a positive value that is very close to our graphical solution. When $t \approx 0.9828$, $s(0.9828) \approx 2\cos(0.9828) + 3\sin(0.9828) \approx 3.6056$. Thus the mass travels about 3.6056 cm from its equilibrium position.

(e) *When is the speed the greatest?*

Speed is the absolute value of the velocity. $|v|$. Since the velocity is zero when the mass reaches its maximum or minimum, the speed will be greatest when the mass is moving through the equilibrium point, *i.e.*, when $s = 0$.

From part (c) and out knowledge of the inverse tangent function, $s = 0$ when $t = \tan^{-1}\left(-\frac{2}{3}\right) + k\pi$, where k is a positive integer since $t > 0$. So the speed will be the greatest when $t \approx 2.5536, 5.6952, 8.8368, \ldots$.

The diagram at right shows the angle $t \approx -0.5880$ that we get from the inverse tangent directly applied to $\tan(t) = -\frac{2}{3}$, and the additional rotation of π that takes us to the next solution of $\tan(t) = -\frac{2}{3}$. For each additional rotation of π, we get another solution to $\tan(t) = -\frac{2}{3}$. Thus we write $t = \tan^{-1}\left(-\frac{2}{3}\right) + k\pi$.