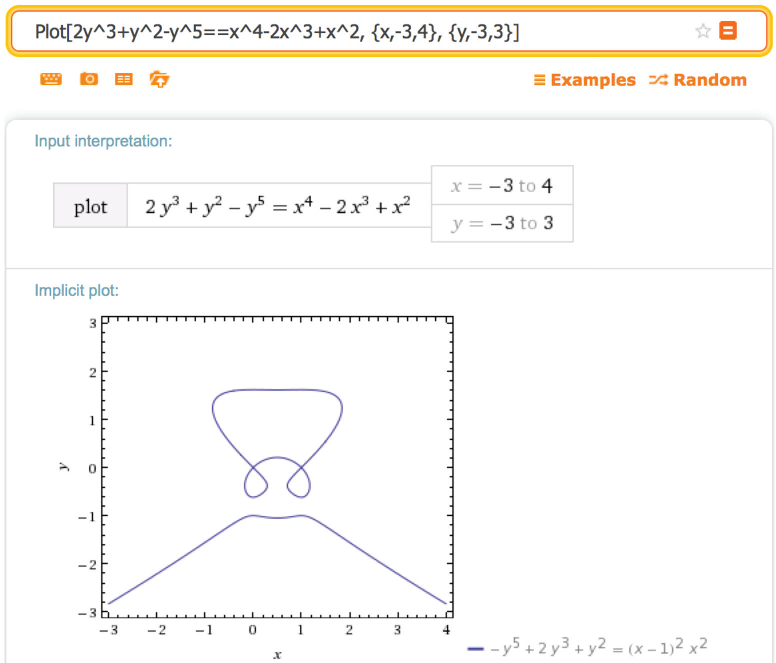
Calculus I, Section 3.5, #42
Implicit Differentiation

(a) *The curve with equation*

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.[1]



(b) *At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.*

From the graph, it seems that there are 6 or 7 points where the tangent lines are horizontal. If the top of the “wagon” is convex, then there are 6; if it is concave then there are 7. Again, these are from the graph, so we could be wrong.

Let’s compute the derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[2y^3 + y^2 - y^5\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^4 - 2x^3 + x^2\right] \\
6y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3 - 6x^2 + 2x \\
\left(6y^2 + 2y - 5y^4\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\left(2x^2 - 3x + 1\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x\,(2x-1)\,(x-1)}{6y^2 + 2y - 5y^4}
\end{aligned}$$

Thus, the slope of the tangent line is 0 when $x = 0$, $x = \frac{1}{2}$, and $x = 1$. For each of these values of x, the corresponding values of y are given by the solutions to

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

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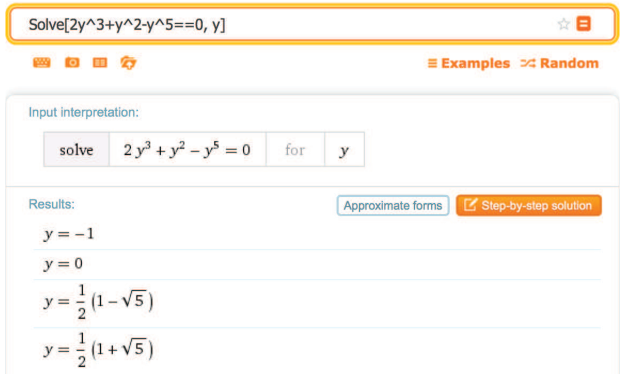
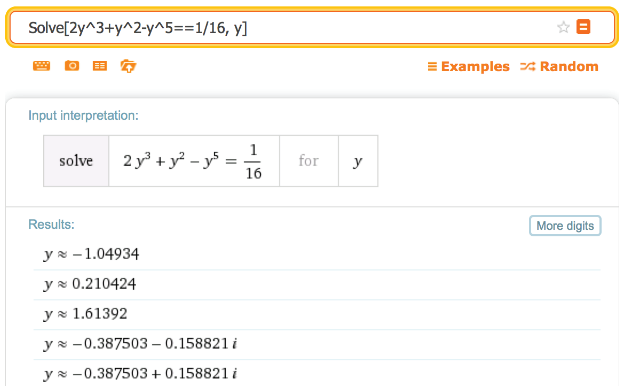
[1] Stewart, *Calculus, Early Transcendentals*, p. 216, #42.

If $x = 0$, we solve $2y^3 + y^2 - y^5 = 0$. WolframAlpha gives us



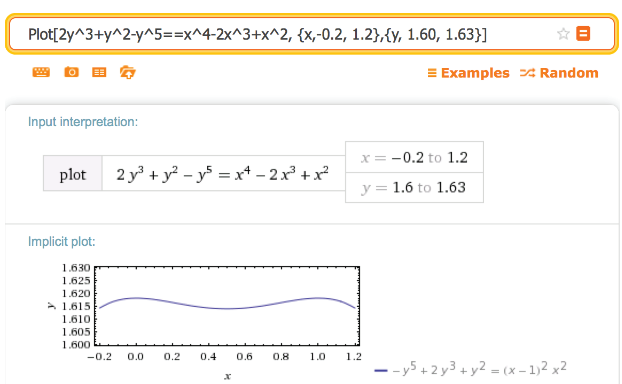
Examining the graph shows us that there is no horizontal tangent at (0,0), so this critical value gives three horizontal tangents. If $x = \frac{1}{2}$, we solve $2y^3 + y^2 - y^5 = \frac{1}{16}$. WolframAlpha gives us



Two of these solutions are complex numbers, so we exclude these.[2] Again, we get three horizontal tangents.

If $x = 1$, we solve $2y^3 + y^2 - y^5 = 0$. As above, we get three horizontal tangents.

Note that our graph does not make it easy to see that there are three critical points along the top of the "wagon." Here is another plot that zooms in on the top of the wagon:



From this plot, we can see there are indeed three horizontal tangents.

Thus, there are 9 horizontal tangents: 3 each at $x = 0$, $x = \frac{1}{2}$, and $x = 1$.

[2] If you wish to study the calculus of complex numbers, major in mathematics, get your B.A., and head off to graduate school.