

Calculus I, Section 3.7, #26
Rates of Change in the Natural and Social Sciences

The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b . According to this model, what happens to the yeast population in the long run?¹

We are told that $n = 20$ when $t = 0$. Let's substitute these values into the function.

$$\begin{aligned}n &= \frac{a}{1 + be^{-0.7t}} \\20 &= \frac{a}{1 + be^{-0.7 \cdot 0}} \\20 &= \frac{a}{1 + b}\end{aligned}$$

so

$$20(1 + b) = a$$

We are also given $\frac{dn}{dt} = 12$ when $t = 0$. Let's compute $\frac{dn}{dt}$ and then substitute.

$$\begin{aligned}n &= \frac{a}{1 + be^{-0.7t}} \\ \frac{dn}{dt} &= \frac{(1 + be^{-0.7t}) \cdot 0 - a \cdot -0.7be^{-0.7t}}{(1 + be^{-0.7t})^2} \\ \frac{dn}{dt} &= \frac{0.7abe^{-0.7t}}{(1 + be^{-0.7t})^2}\end{aligned}$$

Substituting $\frac{dn}{dt} = 12$ when $t = 0$

$$\begin{aligned}12 &= \frac{0.7abe^{-0.7 \cdot 0}}{(1 + be^{-0.7 \cdot 0})^2} \\12 &= \frac{0.7ab}{(1 + b)^2}\end{aligned}$$

Substituting our result for a

$$\begin{aligned}12 &= \frac{0.7 \cdot 20(1 + b) \cdot b}{(1 + b)^2} \\12 &= \frac{14 \cdot b}{(1 + b)} \\ \frac{6}{7} &= \frac{b}{1 + b} \\6(1 + b) &= 7b \\6 &= b\end{aligned}$$

¹Stewart, *Calculus, Early Transcendentals*, p. 235, #26.

and so

$$\begin{aligned}20(1 + 6) &= a \\140 &= a\end{aligned}$$

Thus the specific function is

$$n = f(t) = \frac{140}{1 + 6e^{-0.7t}}$$

“... what happens to the yeast population in the long run?” tells us to examine the limit as $t \rightarrow \infty$.

$$\begin{aligned}\lim_{t \rightarrow \infty} f(t) &= \lim_{t \rightarrow \infty} \frac{140}{1 + 6e^{-0.7t}} \\&= \lim_{t \rightarrow \infty} \frac{140}{1 + \frac{6}{e^{0.7t}}} \\&= \lim_{t \rightarrow \infty} \frac{140}{1} \\&= 140\end{aligned}$$

Thus, in the long run, the population levels off near 140 cells.