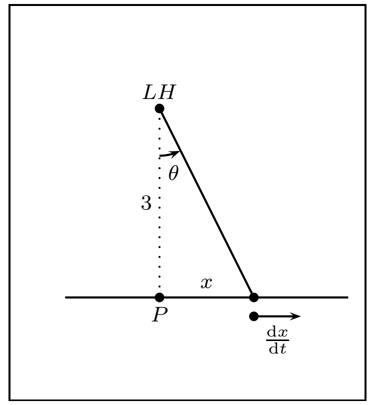
A lighthouse is located on a small island 3 km *away from the nearest point* P *on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is* 1 km *from* P*?*[1]

We know $\frac{d\theta}{dt} = 4\,\text{rev/min} = 4 \cdot 2\pi\,\text{radians/min} = 8\pi\,\text{rads/min}$.

We want $\frac{dx}{dt}$ when $x = 1\,\text{km}$.

We need an equation to relate θ and x. From the diagram at right, we see that

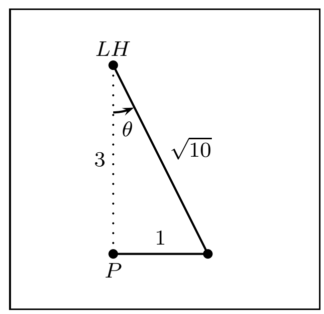
$$\tan(\theta) = \frac{x}{3}$$



and we differentiate with respect to t.

$$\frac{d}{dt}\left[\tan(\theta)\right] = \frac{d}{dt}\left[\frac{x}{3}\right]$$
$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{3} \cdot \frac{dx}{dt}$$
$$3\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$

The second diagram shows the moment when $x = 1$, so $\sec(\theta) = \frac{\sqrt{10}}{3}$, and $\sec^2(\theta) = \frac{10}{9}$. (We found $\sqrt{10}$ with the Pythag. Thm.) Substituting,



$$3 \cdot \frac{10}{9} \cdot 8\pi = \frac{dx}{dt}$$
$$\frac{80\pi}{3} = \frac{dx}{dt}$$
$$83.78 \approx \frac{dx}{dt}$$

Thus, the beam of light is sweeping along the shore at the rate of 83.78 km/min when the beam is 1 km from the point P.

[1] Stewart, *Calculus, Early Transcendentals*, p. 251, #44.