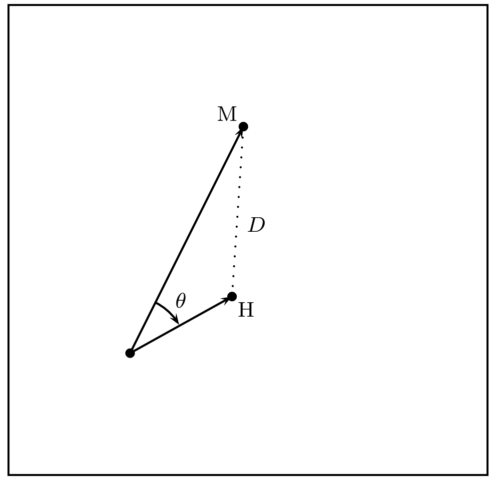
The minute hand on a watch is 8 mm *long and the hour hand is* 4 mm. *How fast is the distance between the tips of the hands changing at one o'clock?*[1]



We know the hour hand travels around the clock at

$$\frac{1\,\text{rev}}{12\,\text{h}} = \frac{2\pi\,\text{radian}}{12\,\text{h}} = \frac{\pi}{6}\frac{\text{rad}}{\text{h}}$$

and the minute hand travels at

$$\frac{1\,\text{rev}}{1\,\text{h}} = \frac{2\pi\,\text{radian}}{1\,\text{h}} = 2\pi\frac{\text{rad}}{\text{h}}$$

so the angle θ between them is changing at

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\pi}{6} - 2\pi = -\frac{11}{6}\pi\frac{\text{rad}}{\text{h}}$$

If we let D be the distance between the tips of the hands, then we want $\frac{\mathrm{d}D}{\mathrm{d}t}$ at one o'clock. At one o'clock, the minute hand is at 12, and the hour hand points to 1, which is $\frac{1}{12}$ of a revolution. So at one o'clock, $\theta = \frac{1}{12}\cdot 2\pi = \frac{\pi}{6}$.

We want $\frac{\mathrm{d}D}{\mathrm{d}t}$ when $\theta = \frac{\pi}{6}$, so we need an equation that relates D and θ. Looking at the diagram, we see that the Law of Cosines gives us

$$D^2 = 4^2 + 8^2 - 2\,(4)\,(8)\cos(\theta)$$
$$D^2 = 80 - 64\cos(\theta)$$

and we differentiate with respect to time t

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[D^2\right] = \frac{\mathrm{d}}{\mathrm{d}t}[80] - \frac{\mathrm{d}}{\mathrm{d}t}[64\cos(\theta)]$$
$$2D\frac{\mathrm{d}D}{\mathrm{d}t} = 0 - 64\cdot -\sin(\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$D\frac{\mathrm{d}D}{\mathrm{d}t} = 32\sin(\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

so

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{32\sin(\theta)}{D}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

When $\theta = \frac{\pi}{6}$, we have

$$D^2 = 80 - 64\cos\left(\frac{\pi}{6}\right)$$
$$D^2 = 80 - 64\cdot\frac{\sqrt{3}}{2}$$
$$D^2 = 80 - 32\sqrt{3}$$

or

$$D = \sqrt{80 - 32\sqrt{3}} \qquad \text{(Since } D \text{ is a distance, we know } D \geq 0\text{)}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 251, #50.

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Finally, we substitute,

$$\begin{aligned}\left.\frac{\mathrm{d}D}{\mathrm{d}t}\right|_{\theta=\pi/6} &= \frac{32\sin(\pi/6)}{\sqrt{80-32\sqrt{3}}}\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ &= \frac{32\cdot\frac{1}{2}}{\sqrt{80-32\sqrt{3}}}\cdot -\frac{11}{6}\pi\\ &= -\frac{88\pi}{3\sqrt{80-32\sqrt{3}}}\\ &\approx -18.59\,\mathrm{mm/h}\end{aligned}$$

or

$$\approx -0.0052\,\mathrm{mm/s}$$

$$\left(-18.59\frac{\mathrm{mm}}{\mathrm{h}}\cdot\frac{1\,\mathrm{h}}{60\,\mathrm{min}}\cdot\frac{1\,\mathrm{min}}{60\,\mathrm{s}} = -0.0052\frac{\mathrm{mm}}{\mathrm{s}}\right)$$