An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is[1]

$$F = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where μ is a positive constant called the coefficient of friction and where $0 \le \theta \le \pi/2$. Show that F is minimized when $\tan(\theta) = \mu$.

We have the closed interval $0 \le \theta \le \pi/2$, so we want the absolute minimum of F on that interval.

$$F = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

so

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}\theta} &= \frac{(\mu \sin(\theta) + \cos(\theta)) \cdot 0 - \mu W(\mu \cos(\theta) - \sin(\theta))}{(\mu \sin(\theta) + \cos(\theta))^2} \\ &= \frac{\mu W \sin(\theta) - \mu^2 W \cos(\theta)}{(\mu \sin(\theta) + \cos(\theta))^2} \\ &= \frac{\mu W(\sin(\theta) - \mu \cos(\theta))}{(\mu \sin(\theta) + \cos(\theta))^2}\end{aligned}$$

and we solve

$$0 = \frac{\mu W(\sin(\theta) - \mu \cos(\theta))}{(\mu \sin(\theta) + \cos(\theta))^2}$$

The denominator is never zero, nor is μW, so we have

$$\begin{aligned}0 &= \sin(\theta) - \mu \cos(\theta) \\ \mu \cos(\theta) &= \sin(\theta) \\ \mu &= \frac{\sin(\theta)}{\cos(\theta)} \\ \mu &= \tan(\theta)\end{aligned}$$

Now we evaluate $F = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$ at the endpoints.

$$\begin{aligned}F(0) &= \frac{\mu W}{\mu \sin(0) + \cos(0)} \\ &= \frac{\mu W}{1} \\ &= \mu W \\ F(\pi/2) &= \frac{\mu W}{\mu \sin(\pi/2) + \cos(\pi/2)} \\ &= \frac{\mu W}{\mu} \\ &= W\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 284, #72.

Continued ⟹

To evaluate F at the critical number, it will help to substitute $\mu = \tan(\theta)$ into the rule for F.

$$\begin{aligned} F &= \frac{\tan(\theta)\, W}{\tan(\theta)\sin(\theta) + \cos(\theta)} \\ &= \frac{W\tan(\theta)}{\frac{\sin(\theta)}{\cos(\theta)}\sin(\theta) + \cos(\theta)} \\ &= \frac{W\tan(\theta)}{\frac{\sin^2(\theta)+\cos^2(\theta)}{\cos(\theta)}} \\ &= \frac{W\tan(\theta)}{1} \cdot \frac{\cos(\theta)}{\sin^2(\theta) + \cos^2(\theta)} \\ &= W\sin(\theta) \end{aligned}$$

Now we need this in terms of our angle θ, so we'll use the right triangle shown below.

Since $\sin(\theta) = \frac{\mu}{\sqrt{\mu^2+1}}$, we get

$$F = \frac{\mu}{\sqrt{\mu^2+1}} W$$

Now, we can see that $\frac{\mu}{\sqrt{\mu^2+1}} < 1$ and $\frac{\mu}{\sqrt{\mu^2+1}} < \mu$ since the denominator is positive and greater than μ, and thus $\frac{\mu}{\sqrt{\mu^2+1}} W < W$ and $\frac{\mu}{\sqrt{\mu^2+1}} W < \mu W$.

Thus, the absolute minimum of $F = \frac{\mu W}{\mu\sin(\theta)+\cos(\theta)}$ is $\frac{\mu}{\sqrt{\mu^2+1}} W$ and this occurs when $\tan(\theta) = \mu$.

Since $\sin(\theta) = \frac{\mu}{\sqrt{\mu^2+1}}$,