For the function[1]

$$f(x) = \cos^2(x) - 2\sin(x), \quad 0 \le x \le 2\pi$$

(a) Find the intervals on which f is increasing or decreasing.

We need to find the intervals where f' is positive and where f' is negative.

$$\begin{aligned} f'(x) &= 2\cos(x) \cdot -\sin(x) - 2\cos(x) \\ &= -2\cos(x)(\sin(x) + 1) \end{aligned}$$

The critical numbers are the solutions to

$$0 = -2\cos(x)(\sin(x) + 1)$$

so

$$\begin{aligned} 0 &= \cos(x) \text{ or } 0 = \sin(x) + 1 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } x = \frac{3\pi}{2} \end{aligned}$$

Now we'll analyze the signs of the factors of f' on the interval $0 \le x \le 2\pi$.[2]

Factor \ Interval	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
-2	$-$	$-$	$-$
$\cos(x)$	$+$	$-$	$+$
$\sin(x) + 1$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$
f	decreasing	increasing	decreasing

Thus the function $f(x) = \cos^2(x) - 2\sin(x)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$, increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and decreasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.

(b) Find the local maximum and minimum values of f.

From the table above, note that the derivative changes sign from $-$ to $+$ at $x = \frac{\pi}{2}$, so by First Derivative Test (FDT) the function has a local minimum of $f\left(\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{\pi}{2}\right) = 0^2 - 2 \cdot 1 = -2$ that occurs at $x = \frac{\pi}{2}$.

The derivative changes sign from $+$ to $-$ at $x = \frac{3\pi}{2}$, so by First Derivative Test (FDT) the function has a local maximum of $f\left(\frac{3\pi}{2}\right) = \cos^2\left(\frac{3\pi}{2}\right) - 2\sin\left(\frac{3\pi}{2}\right) = 0^2 - 2 \cdot -1 = 2$ that occurs at $x = \frac{3\pi}{2}$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 301, #14.

[2] Some teachers prefer that their students use a table and analyze the behavior of the factors of f' to determine the sign of f', whereas others have their students test values in the intervals to determine the sign of f'. In the long-run, its probably best to understand how to analyze the factors, but be sure you know what your teacher wants you to do.

Continued $\Longrightarrow$

(c) Find the intervals of concavity and the inflection points.

We need to find the intervals where f'' is positive and where f'' is negative.

$$f'(x) = -2\cos(x)(\sin(x)+1)$$

so

$$\begin{aligned} f''(x) &= -2\left[\cos(x)\cdot(\cos(x)+0)+(\sin(x)+1)\cdot -\sin(x)\right] \\ &= -2\left[\cos^2(x)-\sin^2(x)-\sin(x)\right] \\ &= -2\left[1-\sin^2(x)-\sin^2(x)-\sin(x)\right] \\ &= -2\left[-2\sin^2(x)-\sin(x)+1\right] \\ &= 2\left[2\sin^2(x)+\sin(x)-1\right] \\ &= 2(2\sin(x)-1)(\sin(x)+1) \end{aligned}$$

The critical numbers of f' are the solutions to

$$0 = 2(2\sin(x)-1)(\sin(x)+1)$$

so

$$\begin{aligned} 0 = 2\sin(x)-1 &\text{ or } 0=\sin(x)+1 \\ \frac{1}{2}=\sin(x) &\text{ or } -1=\sin(x) \\ x=\frac{\pi}{6},\ x=\frac{5\pi}{6} &\text{ or } x=\frac{3\pi}{2} \end{aligned}$$

Now we'll analyze the signs of the factors of f'' on the interval $0 \le x \le 2\pi$.

Factor \ Interval	$\left(0,\frac{\pi}{6}\right)$	$\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2},2\pi\right)$
2	+	+	+	+
$2\sin(x)-1$	−	+	−	−
$\sin(x)+1$	+	+	+	+
$f'(x)$	−	+	−	−
f	concave down	concave up	concave down	concave down

Thus the function $f(x)=\cos^2(x)-2\sin(x)$ is concave down on $\left(0,\frac{\pi}{6}\right)$, concave up on $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$, concave down on $\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$, and concave down on $\left(\frac{3\pi}{2},2\pi\right)$.

The concavity changes at $x=\frac{\pi}{6}$, so there is an infection point at

$$\begin{aligned} \left(\frac{\pi}{6}, f\left(\frac{\pi}{6}\right)\right) &= \left(\frac{\pi}{6}, \cos^2\left(\frac{\pi}{6}\right)-2\sin\left(\frac{\pi}{6}\right)\right) \\ &= \left(\frac{\pi}{6}, \left(\frac{\sqrt{3}}{2}\right)^2 - 2\cdot\frac{1}{2}\right) \\ &= \left(\frac{\pi}{6}, -\frac{1}{4}\right) \end{aligned}$$

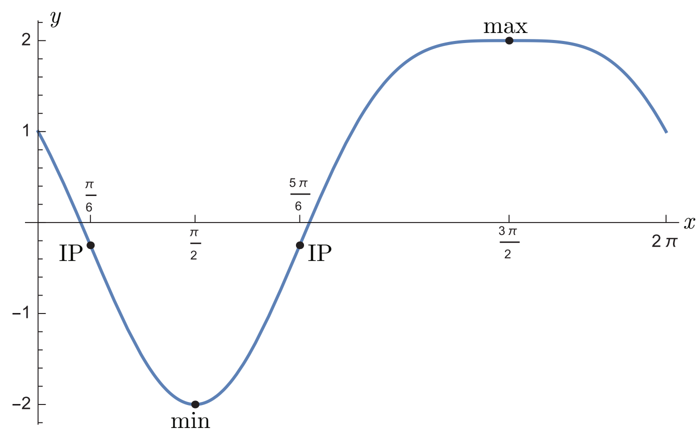
Continued $\Longrightarrow$

The concavity changes at $x = \frac{5\pi}{6}$, so there is an infection point at

$$\begin{aligned}\left(\frac{5\pi}{6}, f\left(\frac{5\pi}{6}\right)\right) &= \left(\frac{5\pi}{6}, \cos^2\left(\frac{5\pi}{6}\right) - 2\sin\left(\frac{5\pi}{6}\right)\right) \\ &= \left(\frac{5\pi}{6}, \left(-\frac{\sqrt{3}}{2}\right)^2 - 2\cdot\frac{1}{2}\right) \\ &= \left(\frac{5\pi}{6}, -\frac{1}{4}\right)\end{aligned}$$

Note that at $x = \frac{3\pi}{2}$, the second derivative is zero, but the sign does not change, so there is *no* inflection point at $x = \frac{3\pi}{2}$.

Here is a graph of our function $f(x) = \cos^2(x) - 2\sin(x), \quad 0 \le x \le 2\pi$.



It is important to note that the portion of the graph on $0 < x < \frac{\pi}{6}$ is not obviously concave down. Even with a nice, large, computer-generated graph such as this, it is difficult to see the concavity. But the calculus and algebra tell us it is there! This is a great example of why we *cannot* rely on graphs to make conclusions about the behavior of functions.