For the function[1]

$$G(x) = 5x^{2/3} - 2x^{5/3}$$

(a) Find the intervals of increase or decrease.

We need to find the intervals where G' is positive and where G' is negative.

$$\begin{aligned} G'(x) &= 5 \cdot \frac{2}{3}x^{-1/3} - 2 \cdot \frac{5}{3}x^{2/3} \\ &= \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} \\ &= \frac{10}{3}x^{-1/3}\,(1-x) \\ &= \frac{10\,(1-x)}{3x^{1/3}} \end{aligned}$$

Note that $G'(x)$ is undefined at $x = 0$. The critical numbers are $x = 0$ and the solutions to

$$\begin{aligned} 0 &= \frac{10\,(1-x)}{3x^{1/3}}, \quad x \neq 0 \\ 0 &= 10\,(1-x) \\ 0 &= 1 - x \\ x &= 1 \end{aligned}$$

Hence, the critical numbers for G are

$$x = 0 \text{ or } x = 1$$

Now we'll analyze the signs of the factors of G'.

Factor \ Interval	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
10	$+$	$+$	$+$
$1-x$	$+$	$+$	$-$
3	$+$	$+$	$+$
$\sqrt[3]{x}$	$-$	$+$	$+$
$G'(x)$	$-$	$+$	$-$
G	decreasing	increasing	decreasing

Thus the function $G(x) = 5x^{2/3} - 2x^{5/3}$ is decreasing on $(-\infty,0)$, increasing on $(0,1)$, and decreasing on $(1,\infty)$.

(b) Find the local maximum and minimum values.

From the table above, note that the derivative changes sign from $-$ to $+$ at $x = 0$, so by First Derivative Test (FDT) the function has a local minimum of $G(0) = 5 \cdot 0^{2/3} - 2 \cdot 0^{5/3} = 0$ that occurs at $x = 0$. (Note that FDT applies even though G' is undefined because G is continuous at $x = 0$.)

The derivative changes sign from $+$ to $-$ at $x = 1$, so by First Derivative Test (FDT) the function has a local minimum of $G(1) = 5 \cdot 1^{2/3} - 2 \cdot 1^{5/3} = 3$ that occurs at $x = 1$.

[1]Stewart, *Calculus, Early Transcendentals*, p. 302, #44.

Continued $\Longrightarrow$

(c) Find the intervals of concavity and the inflection points.

We need to find the intervals where f'' is positive and where f'' is negative.

$$G'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3}$$

so

$$\begin{aligned} G''(x) &= \frac{10}{3} \cdot -\frac{1}{3}x^{-4/3} - \frac{10}{3} \cdot \frac{2}{3}x^{-1/3} \\ &= -\frac{10}{9}x^{-4/3} - \frac{20}{3}x^{-1/3} \\ &= -\frac{10}{9}x^{-4/3}(1+2x) \\ &= -\frac{10(1+2x)}{9x^{4/3}} \end{aligned}$$

Note that G'' is undefined at $x = 0$, so the critical numbers of G' are $x = 0$ and the solutions to

$$\begin{aligned} 0 &= -\frac{10(1+2x)}{9x^{4/3}}, \quad x \neq 0 \\ 0 &= -10(1+2x) \\ 0 &= 1+2x \\ x &= -\frac{1}{2} \end{aligned}$$

so

$$x = 0 \text{ or } x = -\frac{1}{2}$$

Now we'll analyze the signs of the factors of G''.

Factor \ Interval	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, 0\right)$	$(0,\infty)$
-10	$-$	$-$	$-$
$1+2x$	$-$	$+$	$+$
9	$+$	$+$	$+$
$\sqrt[3]{x^4}$	$+$	$+$	$+$
$G''(x)$	$+$	$-$	$-$
G	concave up	concave down	concave down

Thus the function $G(x) =$ is concave up on $\left(-\infty, -\frac{1}{2}\right)$, concave down on $\left(-\frac{1}{2}, 0\right)$, and concave down on $(0,\infty)$.

The concavity changes at $x = -\frac{1}{2}$, so there is an infection point at

$$\begin{aligned} \left(-\frac{1}{2}, G\left(-\frac{1}{2}\right)\right) &= \left(-\frac{1}{2}, 5\left(-\frac{1}{2}\right)^{2/3} - 2\left(-\frac{1}{2}\right)^{5/3}\right) \\ &= \left(-\frac{1}{2}, 5\sqrt[3]{\frac{1}{4}} - 2\sqrt[3]{-\frac{1}{32}}\right) \end{aligned}$$

Continued $\Longrightarrow$

after many steps of simplification

$$= \left(-\frac{1}{2}, 3\sqrt[3]{2}\right)$$

(d) *Use the information from parts (a)-(c) to sketch the graph. Check your work with a graphing device.*