The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\left(2\sigma^2\right)}$$

occurs in probability and statistics, where it is called the ***normal density function****. The constant* μ *is called the* ***mean*** *and the positive constant* σ *is called the* ***standard deviation****. For simplicity, let's scale the function so as to remove the factor* $1/\left(\sigma\sqrt{2\pi}\right)$ *and let's analyze the special case where* $\mu = 0$*. So we study the function*[1]

$$f(x) = e^{-x^2/\left(2\sigma^2\right)}$$

(a) *Find the asymptote, maximum value, and inflection points of* f*.*

Since f is defined by an exponential function, there is no vertical asymptote. The horizontal asymptote is given by the limit of f as $x \to \pm\infty$.

$$\begin{aligned}
\lim_{x\to-\infty} e^{-x^2/\left(2\sigma^2\right)} &= \lim_{x\to-\infty} \left(e^{-x^2}\right)^{\left(2\sigma^2\right)} \\
&= \lim_{x\to-\infty} \left(\frac{1}{e^{x^2}}\right)^{\left(2\sigma^2\right)} \\
&= \left(\lim_{x\to-\infty} \frac{1}{e^{x^2}}\right)^{\left(2\sigma^2\right)}
\end{aligned}$$

As $x \to -\infty$, $x^2 \to \infty$, and $\frac{1}{e^{x^2}} \to 0$. Thus,

$$\begin{aligned}
&= (0)^{\left(2\sigma^2\right)} \\
&= 0
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{x\to\infty} e^{-x^2/\left(2\sigma^2\right)} &= \lim_{x\to\infty} \left(e^{-x^2}\right)^{\left(2\sigma^2\right)} \\
&= \lim_{x\to\infty} \left(\frac{1}{e^{x^2}}\right)^{\left(2\sigma^2\right)} \\
&= \left(\lim_{x\to\infty} \frac{1}{e^{x^2}}\right)^{\left(2\sigma^2\right)}
\end{aligned}$$

As $x \to \infty$, $x^2 \to \infty$, and $\frac{1}{e^{x^2}} \to 0$. Thus,

$$\begin{aligned}
&= (0)^{\left(2\sigma^2\right)} \\
&= 0
\end{aligned}$$

Thus, the horizontal asymptote is $y = 0$, *i.e.*, the x-axis.

[1]Stewart, *Calculus, Early Transcendentals*, p. 303, #72.

Continued $\Longrightarrow$

To find the maximum value, we need to find the critical number(s) for f.

$$f(x) = e^{-x^2/\left(2\sigma^2\right)}$$

so

$$\begin{aligned} f'(x) &= e^{-x^2/\left(2\sigma^2\right)} \cdot \frac{-2x}{2\sigma^2} \\ &= -\frac{x}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)} \end{aligned}$$

and we solve

$$0 = -\frac{x}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)}$$

and since $e^{-x^2/\left(2\sigma^2\right)}$ is never zero and σ^2 is positive, we have

$$\begin{aligned} 0 &= -\frac{x}{\sigma^2} \\ 0 &= x \end{aligned}$$

To determine that this critical value yields a maximum, we use SDT.

$$f'(x) = -\frac{x}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)}$$

so

$$\begin{aligned} f''(x) &= -\frac{x}{\sigma^2} \cdot e^{-x^2/\left(2\sigma^2\right)} \cdot -\frac{x}{\sigma^2} + e^{-x^2/\left(2\sigma^2\right)} \cdot -\frac{1}{\sigma^2} \\ &= -\frac{1}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)}\left(-\frac{x^2}{\sigma^2} + 1\right) \\ &= -\frac{1}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)}\left(\frac{\sigma^2 - x^2}{\sigma^2}\right) \end{aligned}$$

Substituting the critical value $x = 0$,

$$\begin{aligned} f''(0) &= -\frac{1}{\sigma^2}e^{-(0)^2/\left(2\sigma^2\right)}\left(\frac{\sigma^2 - (0)^2}{\sigma^2}\right) \\ &= -\frac{1}{\sigma^2} \cdot 1 \\ &= -\frac{1}{\sigma^2} \end{aligned}$$

Since σ is positive, $f''(0) < 0$. Thus, by SDT, the function $f(x) = e^{-x^2/\left(2\sigma^2\right)}$ has a maximum value of $f(0) = 1$ that occurs at $x = 0$.

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To find the points of inflection, we solve

$$0 = f''(x)$$
$$0 = -\frac{1}{\sigma^2}\mathrm{e}^{-x^2/\left(2\sigma^2\right)}\left(\frac{\sigma^2 - x^2}{\sigma^2}\right)$$

Note that $-\frac{1}{\sigma^2}\mathrm{e}^{-x^2/\left(2\sigma^2\right)}$ is never zero, so we have

$$0 = \frac{\sigma^2 - x^2}{\sigma^2}$$
$$0 = \sigma^2 - x^2$$
$$0 = (\sigma - x)(\sigma + x)$$

so

$$x = \sigma \quad \text{or} \quad x = -\sigma$$

If $x < -\sigma$, then $\sigma^2 - x^2 < 0$, and $f'' > 0$. Similarly, if $x > \sigma$, then $\sigma^2 - x^2 < 0$, and $f'' > 0$. If $-\sigma < x < \sigma$, then $\sigma^2 - x^2 > 0$, and $f'' < 0$.

Thus, there are inflection points at $x = -\sigma$, $y = \mathrm{e}^{-1/2}$, and at $x = \sigma$, $y = \mathrm{e}^{-1/2}$.

(b) *What role does σ play in the shape of the curve?*

$\pm\sigma$ gives the x-coordinates of the inflection points, so as σ increases (or decreases), the inflection points move farther (or closer) to the y-axis.

(c) *Illustrate by graphing four members of this family on the same screen.*