If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}\left(1 - \mathrm{e}^{-ct/m}\right)$$

where g is the acceleration due to gravity and c is the positive constant proportionally relating air resistance to the speed.[1]

(a) *Calculate* $\lim_{t\to\infty} v$. *What is the meaning of this limit?*

$$\lim_{t\to\infty} \frac{mg}{c}\left(1 - \mathrm{e}^{-ct/m}\right)$$
$$= \lim_{t\to\infty} \frac{mg}{c}\left(1 - \frac{1}{\mathrm{e}^{ct/m}}\right)$$

and as $t \to \infty$, $\frac{1}{\mathrm{e}^{ct/m}} \to \infty$ since both c and m are positive, so

$$= \frac{mg}{c}(1-0)$$
$$= \frac{mg}{c}$$

This limit is the speed the object approaches as it falls, *i.e.*, the object's terminal velocity.

(b) *For fixed t, use l'Hospitals's Rule to calculate* $\lim_{c\to 0^+} v$. *What can you conclude about the velocity of a falling object in a vacuum?*

Here, as $c \to 0+$, the effect of air resistance is approaching zero, so the object falls as if in a vacuum.

$$\lim_{c\to 0^+} \frac{mg}{c}\left(1 - \mathrm{e}^{-ct/m}\right)$$
$$= mg \lim_{c\to 0^+} \frac{1 - \mathrm{e}^{-ct/m}}{c}$$

as $c \to 0^+$, $\mathrm{e}^{-ct/m} \to 1$, so

$$= mg \lim_{c\to 0^+} \frac{1 - \mathrm{e}^{-ct/m}}{c} \quad \Longrightarrow \quad \frac{0}{0}$$

Applying l'Hospital's Rule with respect to c,

$$\overset{\mathrm{H}}{=} mg \lim_{c\to 0^+} \frac{0 - \mathrm{e}^{-ct/m} \cdot -\frac{t}{m}}{1}$$
$$= mg \cdot \frac{t}{m}$$
$$= gt$$

Thus, in a vacuum, $v = gt$, and the velocity of the falling object is directly proportional to the time for which it has been falling. That is, in a vacuum, the object's speed continues to increase with out bound.

[1] Stewart, *Calculus, Early Transcendentals*, p. 312, #78.