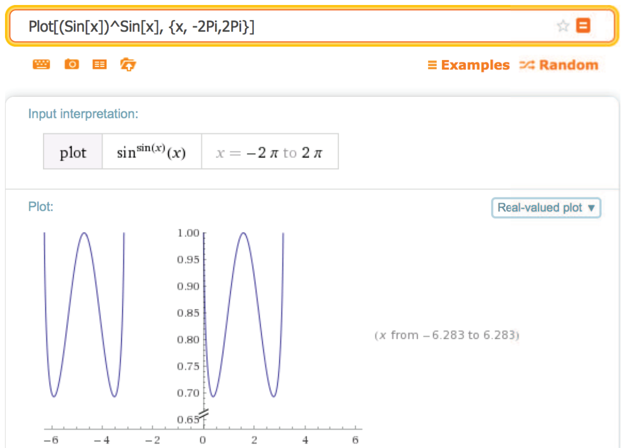
Calculus I, Section 4.6, #26
Graphing with Calculus and Calculators

(a) *Graph the function.*[1]

$$f(x) = (\sin(x))^{\sin(x)}$$

WolframAlpha gives



Since the exponential function $f(x) = (\sin(x))^{\sin(x)}$ is only defined when the base $\sin(x)$ is positive, the computer is only graphing on intervals like $(-2\pi, -\pi)$, $(0,\pi)$, $(2\pi,3\pi)$, *i.e.*, $(2k\pi, (2k+1)\pi)$, where k is an integer.

(b) *Explain the shape of the graph by computing the limit as $x \to 0^+$ or as $x \to \infty$.* Clearly, the limit as $x \to \infty$ does not exist because of the oscillation of the function.

We have

$$y = (\sin(x))^{\sin(x)}$$

so

$$\begin{aligned} \ln(y) &= \ln\left((\sin(x))^{\sin(x)}\right) \\ \ln(y) &= \sin(x) \cdot \ln(\sin(x)) \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to 0^+} \ln(y) &= \lim_{x\to 0^+} \sin(x) \cdot \ln(\sin(x)) && \text{The limit on the right has the form } 0 \cdot -\infty \\ &= \lim_{x\to 0^+} \frac{\ln(\sin(x))}{\csc(x)} && \text{The limit on the right has the form } \frac{-\infty}{\infty} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{\sin(x)} \cdot \cos(x)}{-\csc(x)\cot(x)} \\ &= \lim_{x\to 0^+} \frac{\cot(x)}{-\csc(x)\cot(x)} \\ &= \lim_{x\to 0^+} \frac{1}{-\csc(x)} \\ &= \lim_{x\to 0^+} -\sin(x) \\ &= 0 \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 329, #26.

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Now we have

$$\lim_{x\to 0^+} \ln(y) = 0$$

so

$$\begin{aligned} \ln\left(\lim_{x\to 0^+} y\right) &= 0 \\ \lim_{x\to 0^+} y &= \mathrm{e}^0 \\ &= 1 \end{aligned}$$

Thus, $\lim_{x\to 0^+} (\sin(x))^{\sin(x)} = 1$ as it seems from the graph.

(c) *Estimate the maximum and minimum values and then use calculus to find the exact values.*

Using WolframAlpha (or the TI-84), the local maximum seems to be 1, and the local minimum seems to be ≈ 0.6922

We have

$$\ln(y) = \sin(x) \cdot \ln(\sin(x))$$

so

$$\begin{aligned} \frac{1}{y} \cdot y' &= \sin(x) \cdot \frac{1}{\sin(x)} \cdot \cos(x) + \ln(\sin(x)) \cdot \cos(x) \\ &= \cos(x)\left(1 + \ln(\sin(x))\right) \\ y' &= y \cdot \cos(x)\left(1 + \ln(\sin(x))\right) \\ y' &= (\sin(x))^{\sin(x)} \cos(x)\left(1 + \ln(\sin(x))\right) \end{aligned}$$

Since $(\sin(x))^{\sin(x)}$ is never zero, $y' = 0$ when $\cos(x) = 0$ or when $1 + \ln(\sin(x)) = 0$. When $\cos(x) = 0$, $x = \frac{\pi}{2}$ and when $1 + \ln(\sin(x)) = 0 \implies \ln(\sin(x)) = -1 \implies \sin(x) = \mathrm{e}^{-1} \implies x = \sin^{-1}\left(\mathrm{e}^{-1}\right)$. Using the TI-84 and our knowledge of inverse trig functions, we get three critical points on the interval $(0,\pi)$: $x \approx 0.3767$, $x = \frac{\pi}{2}$, and $x \approx 2.7649$.

Thus, there is a local minimum of ≈ 0.6922 that occurs at $x \approx 0.3767$ and ≈ 2.7649 and a local maximum of 1 that occurs at $x = \frac{\pi}{2}$.

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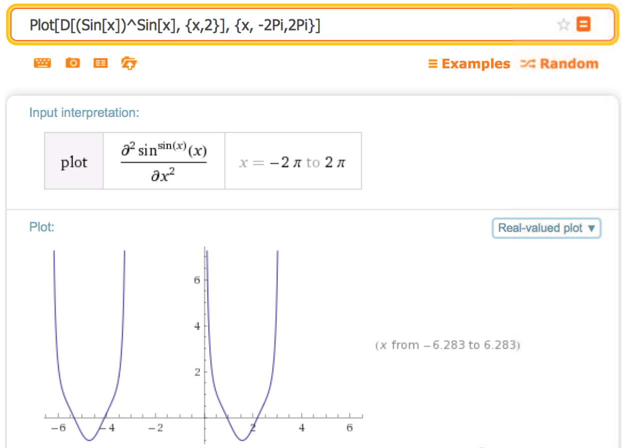
(d) *Use a graph of f'' to estimate the x-coordinates of the inflection points.*

WolframAlpha gives us



The inflection points occur where f'' changes sign, so $x \approx 1$ and $x \approx 2.3$.