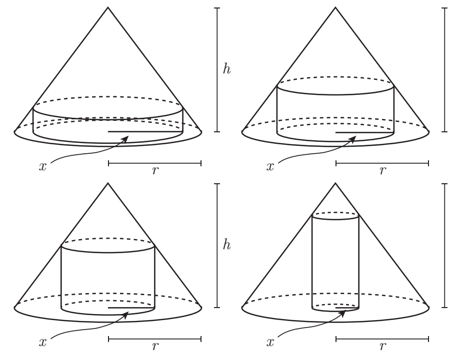
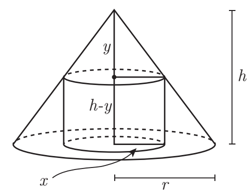
A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cone.[1]



At right are four sketches of various cylinders inscribed a cone of height h and radius r. From these sketches, it seems that the volume of the cylinder changes as a function of the cylinder's radius, x.

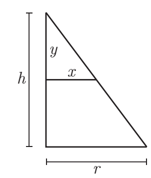
So we need a function that gives the volume of the cylinder in terms of x.

$$\text{Volume}_{cylinder} = (\text{area of base}_{cylinder})\left(\text{height}_{cylinder}\right)$$
$$\text{Volume}_{cylinder} = \left(\pi x^2\right)\left(\text{height}_{cylinder}\right)$$



In the diagram at right, we'll let

$$y = \text{height between top of cylinder and top of cone}$$
$$h - y = \text{height of the cylinder}$$



Finally, we'll use the diagram of the similar triangles at right to find the height of the cylinder in terms of x.

$$\frac{y}{x} = \frac{h}{r}$$
$$\frac{y}{x} \cdot x = \frac{h}{r} \cdot x$$
$$y = \frac{h}{r}x$$

If we substitute this value into our volume function, we get

$$\begin{aligned}\text{Volume}_{cylinder} &= \left(\pi x^2\right)\left(\text{height}_{cylinder}\right)\\ &= \left(\pi x^2\right)(h-y)\\ &= \left(\pi x^2\right)\left(h - \frac{h}{r}x\right)\\ &= \pi x^2 h - \frac{\pi x^3 h}{r}\end{aligned}$$

so

$$V(x) = \pi x^2 h - \frac{\pi h}{r}x^3$$

Now we compute the derivative, find the critical points and determine if the critical points give a local maximum or local minimum.

[1] Stewart, *Calculus, Early Transcendentals*, p. 338, #32.

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$$V(x) = \pi x^2 h - \frac{\pi h}{r} x^3$$

so

$$\begin{aligned} V'(x) &= 2\pi x h - \frac{3\pi h}{r} x^2 \\ &= \pi x h \left(2 - \frac{3}{r} x\right) \end{aligned}$$

so

$$x = 0 \quad \text{or} \quad x = \frac{2}{3} r$$

Clearly, $x = 0$ does not give a maximum volume, so we test $\frac{2}{3}r$

$$V''(x) = 2\pi h - \frac{6\pi h}{r} x$$

and if $x = \frac{2}{3}r$,

$$\begin{aligned} V''\left(\frac{2}{3}r\right) &= 2\pi h - \frac{6\pi h}{r} \cdot \frac{2}{3} r \\ &= 2\pi h - 4\pi h \\ &= -2\pi \\ &< 0 \end{aligned}$$

so by the second derivative test, the critical value $x = \frac{2}{3}r$ gives a maximum volume of

$$\begin{aligned} V\left(\frac{2}{3}r\right) &= \pi \left(\frac{2}{3}r\right)^2 h - \frac{\pi h}{r} \left(\frac{2}{3}r\right)^3 \\ &= \pi \left(\frac{4}{9}r^2\right) h - \pi \frac{h}{r} \left(\frac{8}{27}r^3\right) \\ &= \pi \frac{12}{27} r^2 h - \pi \frac{8}{27} r^2 h \\ &= \frac{4}{27} \pi r^2 h \end{aligned}$$

Thus the maximum volume of a cylinder inscribed in a cone of radius r and height h is $\frac{4}{27}\pi r^2 h$.