For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v-u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v-u}$$

where a is the proportionality constant.[1]

(a) *Determine the value v that minimizes E.*

$$\begin{aligned} E'(v) &= \frac{(v-u)\cdot 3v^2aL - av^3L\cdot 1}{(v-u)^2} \\ &= \frac{aLv^2\left(3\left(v-u\right)-v\right)}{(v-u)^2} \\ &= \frac{aLv^2\left(2v-3u\right)}{(v-u)^2} \end{aligned}$$

Note that $E'(v)$ is undefined if $v = u$. So

$$E'(v) = 0$$

when

$$v = 0 \quad \text{or} \quad v = \frac{3u}{2}$$

Here, $v = 0$ is out of the domian because it implies that the fish is not swimming at all, so the only critical number is $v = \frac{3u}{2}$.

$$\begin{aligned} E''(v) &= \frac{(v-u)^2\left(aLv^2\cdot 2 + (2v-3u)\cdot 2aLv\right) - aLv^2\left(2v-3u\right)\cdot 2\left(v-u\right)\cdot 1}{\left((v-u)^2\right)^2} \\ &= \frac{(v-u)^2\left(6aLv^2 - 6aLvu\right) - 2aLv^2\left(2u-3v\right)\left(v-u\right)}{(v-u)^4} \\ &= \frac{(v-u)^2\left(v-u\right)\left(6aLv\right) - 2aLv^2\left(2u-3v\right)\left(v-u\right)}{(v-u)^4} \\ &= \frac{(v-u)^2\cdot 6aLv - 2aLv^2\left(2v-3u\right)}{(v-u)^3} \\ &= \frac{\left(v^2-2vu+u^2\right)\cdot 6aLv - 4aLv^3 + 6aLv^2u}{(v-u)^3} \\ &= \frac{6aLv^3 - 12aLv^2u + 6aLvu^2 - 4aLv^3 + 6aLv^2u}{(v-u)^3} \\ &= \frac{2aLv^3 - 6aLv^2u + 6aLvu^2}{(v-u)^3} \\ &= \frac{2aLv\left(v^2-3vu+3u^2\right)}{(v-u)^3} \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 338, #46.

Continued ⟹

Now we substitute our critical number into the second derivative.

$$\begin{aligned}
E''(v) &= \frac{2aLv\left(v^2-3vu+3u^2\right)}{(v-u)^3} \\
E''\left(\frac{3u}{2}\right) &= \frac{2aL\cdot\frac{3u}{2}\left(\left(\frac{3u}{2}\right)^2-3\cdot\frac{3u}{2}\cdot u+3u^2\right)}{\left(\frac{3u}{2}-u\right)^3} \\
&= \frac{3aLu\left(\frac{9u^2}{4}-\frac{9u^2}{2}+3u^2\right)}{\left(\frac{3u-2u}{2}\right)^3} \\
&= \frac{3aLu\left(\frac{9u^2}{4}-\frac{18u^2}{4}+\frac{12u^2}{4}\right)}{\left(\frac{u}{2}\right)^3} \\
&= \frac{3aLu\left(\frac{3u^2}{4}\right)}{\frac{u^3}{8}} \\
&= \frac{9aLu^3}{4}\cdot\frac{8}{u^3} \\
&= 18aL
\end{aligned}$$

We know $L > 0$ because it is a distance, and we know $a > 0$ because from the original function the energy must be positive, so $E''\left(\frac{3u}{2}\right) = 18aL > 0$.

Thus, by the SDT, the critical number $v = \frac{3u}{2}$ gives a local minimum for the energy.

(b) *Sketch the graph of E.*

There is a vartical asymptote at $v = u$ and a minimum at $v = \frac{3u}{2}$. Also, since $E''(v) > 0$ for all v, the graph is concave up for all v in the domain of $v > u$.