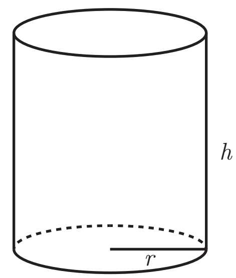
You are given the task of designing a cylindrical tank with a capacity of $40\,\mathrm{m}^3$*. The material for the top and bottom of the tank costs* $\$20\,/\mathrm{m}^2$ *and for the side costs* $\$10\,/\mathrm{m}^2$*. Find the dimensions of the tank that minimizes the cost of materials.*[1]

We need a function C that gives the cost of materials. If we let the radius of the top and bottom be r, then the area of the top and bottom is $2 \cdot \pi r^2$. If we let the height of the tank be h, then the area of the side of the tank is $2\pi rh$.



Thus the cost as a function of r and h is

$$\begin{aligned} C(r,h) &= 20\left(2 \cdot \pi r^2\right) + 10\left(2\pi rh\right) \\ &= 40\pi r^2 + 20\pi rh \end{aligned}$$

We also know the volume of the tank. So

$$\begin{aligned} V &= \pi r^2 h \\ 40 &= \pi r^2 h \end{aligned}$$

so

$$h = \frac{40}{\pi r^2}$$

Substituting into our cost function gives

$$\begin{aligned} C(r) &= 40\pi r^2 + 20\pi r \cdot \frac{40}{\pi r^2} \\ &= 40\pi r^2 + \frac{800}{r} \end{aligned}$$

so

$$C'(r) = 80\pi r - \frac{800}{r^2}$$

Now we find the critical number(s) by solving $C'(r) = 0$.

$$0 = 80\pi r - \frac{800}{r^2}$$

and since $r > 0$,

$$\begin{aligned} 0 &= 80\pi r^3 - 800 \\ 80\pi r^3 &= 800 \\ r^3 &= \frac{10}{\pi} \\ r &= \sqrt[3]{\frac{10}{\pi}} \end{aligned}$$

[1] Suggested by Dr. Anne Hauscarriague, Santiago Canyon College, December 2015.

Continued $\Longrightarrow$

Now $C''(r) = 80\pi + \frac{1600}{r^3}$ and since $r > 0$, $C''(r) > 0$, so by SDT, the critical number $r = \sqrt[3]{\frac{10}{\pi}}$ gives a minimum cost.

The minimum cost is

$$\begin{aligned} C\left(\sqrt[3]{\frac{10}{\pi}}\right) &= 40\pi\left(\sqrt[3]{\frac{10}{\pi}}\right)^2 + \frac{800}{\sqrt[3]{\frac{10}{\pi}}} \\ &\approx 815.76 \end{aligned}$$

The diameter of the tank will be $2 \cdot \sqrt[3]{\frac{10}{\pi}} \approx 2.94\,\mathrm{m}$ and the height of the tank will be $\frac{40}{\pi\left(\sqrt[3]{\frac{10}{\pi}}\right)^2} \approx 5.88\,\mathrm{m}$ and the minimum cost is \$815.76.