

Calculus I, Section 5.1, #31  
Areas and Distances

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Find the exact area under the cosine curve  $y = \cos(x)$  from  $x = a$  to  $x = b$ , where  $0 \leq b \leq \frac{\pi}{2}$ . (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if  $b = \frac{\pi}{2}$ ?

Here we have  $y = f(x) = \cos(x)$ . Also,  $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ . We'll use right endpoints (they're usually simpler) so  $x_i = 0 + i\Delta x = 0 + i \cdot \frac{b}{n} = \frac{bi}{n}$ .

Our equation for area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (\cos(x_i^*) \Delta x) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} \right) \end{aligned}$$

Using WolframAlpha to evaluate the sum, we get

Sum[Cos[b\*i/n]\*b/n, {i,1,n}]

Sum:

$$\sum_{i=1}^n \frac{b \cos\left(\frac{bi}{n}\right)}{n} = \frac{b \left( \sin(b) \cot\left(\frac{b}{2n}\right) + \cos(b) - 1 \right)}{2n}$$

cot(x) is the cotangent function

and then the limit

Limit[Sum[Cos[b\*i/n]\*b/n, {i,1,n}], n->Infinity]

Limit:

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^n \frac{\cos\left(\frac{bi}{n}\right) b}{n} \right) = \sin(b)$$

[Step-by-step solution](#)

Thus

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} \right) = \sin(b)$$

and in particular, if  $b = \frac{\pi}{2}$  the area under the cosine curve from  $x = 0$  to  $x = \frac{\pi}{2}$  is  $\sin\left(\frac{\pi}{2}\right) = 1$ .

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<sup>1</sup>Stewart, *Calculus, Early Transcendentals*, p. 378, #31.