Use the form of the definition of the definite integral given in Theorem 4 to evaluate the integral.[1]

$$\int_0^2 2x - x^3 \, \mathrm{d}x$$

Theorem 4
If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{where} \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x$$

$$\begin{aligned}
\int_0^2 2x - x^3\ \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \qquad \Delta x = \frac{2-0}{n} = \frac{2}{n} \quad \text{and} \quad x_i = 0 + i\Delta x = i\cdot\frac{2}{n} = \frac{2i}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\cdot\frac{2i}{n} - \left(\frac{2i}{n}\right)^3\right)\cdot\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n} - \frac{8i^3}{n^3}\right)\cdot\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n^2} - \frac{16i^3}{n^4}\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{8i}{n^2} - \sum_{i=1}^{n} \frac{16i^3}{n^4}\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n^2}\sum_{i=1}^{n} i - \frac{16}{n^4}\sum_{i=1}^{n} i^3\right)
\end{aligned}$$

and since we know $\sum_{i=1}^{n} i = \frac{n(n+1)}{n}$ and $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{n}\right]^2$,

$$\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{8}{n^2}\cdot\frac{n\,(n+1)}{n} - \frac{16}{n^4}\cdot\left[\frac{n\,(n+1)}{2}\right]^2\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\,(n+1) - \frac{16}{n^4}\cdot\frac{n^2\,(n+1)^2}{4}\right) \\
&= \lim_{n\to\infty} \left(4 + \frac{1}{n} - \frac{4}{n^2}\cdot\left(n^2+2n+1\right)\right) \\
&= \lim_{n\to\infty} \left(4 + \frac{1}{n} - 4 - \frac{8}{n} - \frac{4}{n^2}\right) \\
&= \lim_{n\to\infty} \left(-\frac{7}{n} - \frac{4}{n^2}\right) \\
&= -0 - 0 \\
&= 0
\end{aligned}$$

Thus $\int_0^2 2x - x^3\ \mathrm{d}x = 0$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 389, #24.