Evaluate the integral.[1]

$$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\ \mathrm{d}x$$

We'd like to evaluate this integral using the Fundamental Theorem of Calculus, Part 2 (the option is to compute the limit of the sum).

Let's remind ourselves of the Fundamental Theorem of Calculus, Part 2:

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

First, we need an antiderivative for the function $\frac{4}{\sqrt{1-x^2}}$. This almost fits the basic pattern for $\sin^{-1}(x)$, and if apply one of the properties of the definite integral, we get

$$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} = 4\int_{1/2}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}$$

and the integral on the right does match the basic pattern for $\sin^{-1}(x)$, so

$$\begin{aligned} &= 4\left[\sin^{-1}(x)\right]_{x=1/2}^{x=1/\sqrt{2}} \\ &= 4\cdot\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right] \end{aligned}$$

Now, $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so in a more familiar form,

$$= 4\cdot\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$$

and $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ is the angle whose sine is $\frac{\sqrt{2}}{2}$ or $\frac{\pi}{4}$. Similarly, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{4}$, so we have

$$\begin{aligned} &= 4\cdot\left[\frac{\pi}{4} - \frac{\pi}{6}\right] \\ &= 4\cdot\frac{\pi}{12} \\ &= \frac{\pi}{3} \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 400, #42.