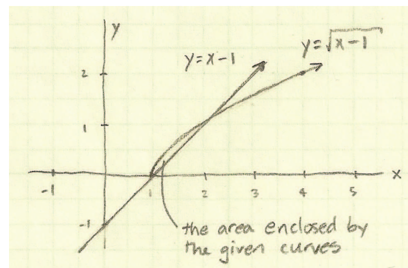
Sketch the region enclosed by the given curves and find its area.[1]

$$y = \sqrt{x-1}, \quad x - y = 1$$

The equation $y = \sqrt{x-1}$ gives y as a function of x, so we'll rewrite the equation $x - y = 1$ with the same relationship between y and x.

$$\begin{aligned} x - y &= 1 \\ -y &= -x + 1 \\ -1 \cdot -y &= -1 \cdot (-x+1) \\ y &= x - 1 \end{aligned}$$

To sketch the region, we recognize $y = \sqrt{x-1}$ as a horizontal translation of $y = \sqrt{x}$ one unit in the positive direction (one unit right), and $y = x - 1$ as a linear function with slope 1 and y-intercept -1.



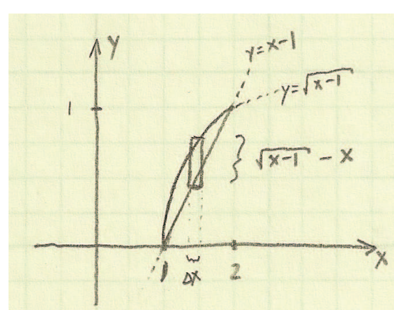
The graphs seem to intersect at (1,0) and (2,1), but we must solve the system of equations to be certain. Substituting $y = x - 1$ into $y = \sqrt{x-1}$ gives us

$$\begin{aligned} x - 1 &= \sqrt{x-1} \\ (x-1)^2 &= \left(\sqrt{x-1}\right)^2 \\ x^2 - 2x + 1 &= x - 1 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \end{aligned}$$

so, from the zero product property,

$$\begin{aligned} x - 1 = 0 \quad &\text{or} \quad x - 2 = 0 \\ x = 1 \quad &\text{or} \quad x = 2 \end{aligned}$$

In the diagram at right, we've drawn a representative rectangle. The base of the representative rectangle is Δx and the height is $\sqrt{x-1} - (x-1)$, so the area is $\left(\sqrt{x-1} - (x-1)\right)\Delta x$. We generate representative rectangles from $x = 1$ to $x = 2$, so the area is given by



$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{x-1} - (x-1)\right)\Delta x$$

$$= \int_1^2 \sqrt{x-1} - (x-1)\ \mathrm{d}x$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 434, #18.

Continued $\Longrightarrow$

$$\int_1^2 \sqrt{x-1} - (x-1) \, \mathrm{d}x$$
$$= \int_1^2 \sqrt{x-1} \, \mathrm{d}x - \int_1^2 x - 1 \, \mathrm{d}x$$

For the left integral, we substitute $u = x - 1$, so $\mathrm{d}u = \mathrm{d}x$. When $x = 1$, $u = 0$, and when $x = 2$, $u = 1$. We can compute the right integral directly.

$$= \int_{u=0}^{u=1} \sqrt{u} \, \mathrm{d}u - \int_{x=1}^{x=2} x - 1 \, \mathrm{d}x$$
$$= \int_{u=0}^{u=1} u^{1/2} \, \mathrm{d}u - \int_{x=1}^{x=2} x - 1 \, \mathrm{d}x$$
$$= \left[\frac{u^{1/2+1}}{1/2+1}\right]_{u=0}^{u=1} - \left[\frac{x^2}{2} - x\right]_{x=1}^{x=2}$$
$$= \left[\frac{u^{3/2}}{3/2}\right]_{u=0}^{u=1} - \left[\frac{x^2}{2} - x\right]_{x=1}^{x=2}$$
$$= \left[\frac{2u^{3/2}}{3}\right]_{u=0}^{u=1} - \left[\frac{x^2}{2} - x\right]_{x=1}^{x=2}$$
$$= \left[\frac{2\,(1)^{3/2}}{3} - \frac{2\,(0)^{3/2}}{3}\right] - \left[\left(\frac{(2)^2}{2} - 2\right) - \left(\frac{(1)^2}{2} - 1\right)\right]$$
$$= \left[\frac{2}{3} - 0\right] - \left[\left(\frac{4}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right]$$
$$= \frac{2}{3} - \left[0 - -\frac{1}{2}\right]$$
$$= \frac{2}{3} - 0 - \frac{1}{2}$$
$$= \frac{1}{6}$$

Thus the area enclosed by the curves $y = \sqrt{x-1}$ and $x - y = 1$ is exactly $\frac{1}{6}$.