Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.[1]

$$x = y^2, \quad x = 1 - y^2; \quad \textit{about } x = 3$$

Let's sketch the graph so we have an idea of what we are working with.

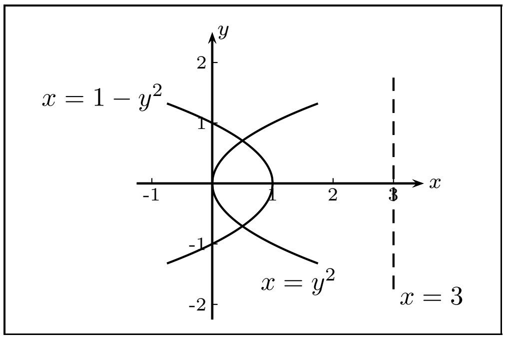

Figure 1

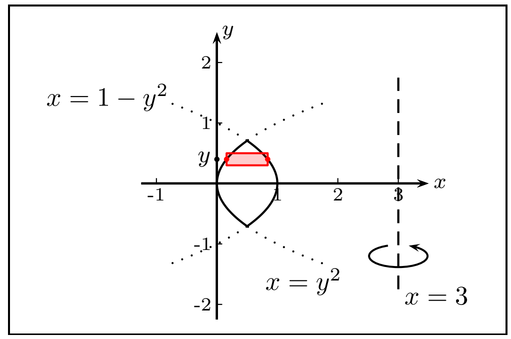

Figure 2

In Figure 1, we've graphed the relations $x = y^2$, $x = 1 - y^2$, and the line $x = 3$ and can see the region bounded by those relations. To find where the graphs intersect, we solve

$$\begin{aligned} y^2 &= 1 - y^2 \\ 2y^2 &= 1 \\ y^2 &= \frac{1}{2} \end{aligned}$$

and from the square root property,

$$\begin{aligned} y &= \sqrt{\frac{1}{2}} \quad \text{or} \quad y = -\sqrt{\frac{1}{2}} \\ y &= \frac{\sqrt{2}}{2} \quad \text{or} \quad y = -\frac{\sqrt{2}}{2} \end{aligned}$$

The graphs intersect at $y = -\frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$, so those are the bounds on our region.

Figure 2 shows a representative rectangle of width Δy sketched in the region, perpendicular to the axis of rotation. We will rotate that representative rectangle about the line $x = 3$, determine the volume of the representative washer, and then add up all the washers from $y = -\frac{\sqrt{2}}{2}$ to $y = \frac{\sqrt{2}}{2}$.

In Figure 3, the washer corresponding to the representative is drawn above. The volume V_w of the washer is given by

$$V_w = (\text{area of the base})\, \Delta y$$

where the base of the washer is the region formed by the two concentric circles.

$$\begin{aligned} V_w &= (\text{area of outside circle} - \text{area of inside circle})\, \Delta y \\ &= \left(\pi r_{out}^2 - \pi r_{in}^2\right) \Delta y \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 446, #17.

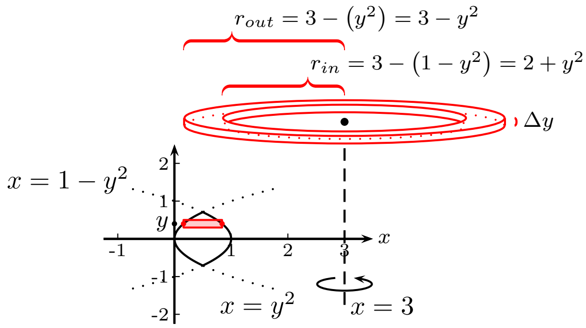

Figure 3

Note that the outside radius r_{out} = *larger value* − *smaller value*, that is, $r_{out} = 3 - \left(y^2\right) = 3 - y^2$ and $r_{in} = 3 - \left(1 - y^2\right) = 2 + y^2$. So

$$V_w = \left(\pi\left(3 - y^2\right)^2 - \pi\left(2 + y^2\right)^2\right)\Delta y$$

We create washers from $y = -\frac{\sqrt{2}}{2}$ to $y = \frac{\sqrt{2}}{2}$, so

$$\begin{aligned}
\text{Volume}_{SOR} &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\pi\left(3 - y^2\right)^2 - \pi\left(2 + y^2\right)^2\right)\Delta y \\
&= \int_{y=-\sqrt{2}/2}^{y=\sqrt{2}/2}\left(\pi\left(3 - y^2\right)^2 - \pi\left(2 + y^2\right)^2\right)\,\mathrm{d}y \\
&= \int_{y=-\sqrt{2}/2}^{y=\sqrt{2}/2}\pi\left(\left(3 - y^2\right)^2 - \left(2 + y^2\right)^2\right)\,\mathrm{d}x \\
&= \pi\int_{y=-\sqrt{2}/2}^{y=\sqrt{2}/2}\left(9 - 6y^2 + y^4\right) - \left(4 + 4y^2 + y^4\right)\,\mathrm{d}y \\
&= \pi\int_{y=-\sqrt{2}/2}^{y=\sqrt{2}/2} -10y^2 + 5\,\mathrm{d}y
\end{aligned}$$

We find an antiderivative for the integrand, and apply the Fundamental Theorem of Calculus.

$$
\begin{aligned}
&= \pi \left[-10 \cdot \frac{y^3}{3} + 5y \right]_{y=-\sqrt{2}/2}^{y=\sqrt{2}/2} \\
&= \pi \left[\left(-10 \cdot \frac{\left(\sqrt{2}/2\right)^3}{3} + 5 \cdot \frac{\sqrt{2}}{2} \right) - \left(-10 \cdot \frac{\left(-\sqrt{2}/2\right)^3}{3} + 5 \cdot -\frac{\sqrt{2}}{2} \right) \right] \\
&= \pi \left[\left(-10 \cdot \frac{2\sqrt{2}/8}{3} + \frac{5\sqrt{2}}{2} \right) - \left(-10 \cdot \frac{-2\sqrt{2}/8}{3} - \frac{5\sqrt{2}}{2} \right) \right] \\
&= \pi \left[\left(-10 \cdot \frac{\sqrt{2}}{12} + \frac{5\sqrt{2}}{2} \right) - \left(-10 \cdot \frac{-\sqrt{2}}{12} - \frac{5\sqrt{2}}{2} \right) \right] \\
&= \pi \left[\left(\frac{-10\sqrt{2}}{12} + \frac{30\sqrt{2}}{12} \right) - \left(\frac{10\sqrt{2}}{12} - \frac{30\sqrt{2}}{12} \right) \right] \\
&= \pi \left[\left(\frac{20\sqrt{2}}{12} \right) - \left(-\frac{20\sqrt{2}}{12} \right) \right] \\
&= \pi \left[\frac{40\sqrt{2}}{12} \right] \\
&= \frac{10\pi\sqrt{2}}{3}
\end{aligned}
$$

Thus the volume of the solid obtained by rotating the region bounded by the graphs of $x = y^2$ and $x = 1 - y^2$ about the line $x = 3$ is $\frac{10\pi\sqrt{2}}{3}$ units^3.