Find the volume of the frustum of a right circular cone with height $h$, lower base radius $R$, and top radius $r$.\(^1\)

A frustum of a cone is the part of the cone that remains after the top of the cone is cut-off parallel to the base of the cone. See Figure 1.

Since the frustum has rotational symmetry, let’s set it up so a representative rectangle is rotated about the $x$-axis.

The slope of the line is $\frac{r-R}{h-0} = \frac{r-R}{h}$, and the $y$-intercept is $R$. Using slope-intercept form, the equation of the line that represents the side of the frustum is $y = \frac{r-R}{h}x + R$.

In Figure 3, we’ve sketched in the disk (washer with no hole) that is generated from revolving the representative rectangle about the $x$-axis. We have

$$r_{\text{out}} = \frac{r-R}{h}x + R$$

$$r_{\text{in}} = 0$$

so the volume of the disk is

$$V_{\text{disk}} = \left(\pi \left(\frac{r-R}{h}x + R\right)^2 - \pi 0^2\right) \Delta x$$

$$= \pi \left(\frac{(r-R)^2}{h^2}x^2 + 2R \cdot \frac{r-R}{h}x + R^2\right) \Delta x$$

\(^1\)Stewart, *Calculus, Early Transcendentals*, p. 447, #48.
Calculus II
Volumes

We create disks from $x = 0$ to $x = h$, so the volume of the solid of revolution, i.e., the volume of the frustum, is given by

$$V_{frustum} = \int_{x=0}^{x=h} \pi \left( \frac{(r-R)^2}{h^2} x^2 + \frac{r-R}{h} x + R^2 \right) dx$$

$$= \pi \left[ \frac{(r-R)^2}{h^2} \cdot \frac{x^3}{3} + \frac{r-R}{h} \cdot \frac{x^2}{2} + R^2 x \right]^{x=h}_{x=0}$$

The integrand looks dreadfully complicated, but keep in mind that all of the coefficients with $r$, $R$, and $h$ are constants and the integration is with respect to $x$. We find an antiderivative for the integrand and apply the Fundamental Theorem of Calculus.

$$= \pi \left[ \frac{(r-R)^2}{h^2} \cdot \frac{h^3}{3} + \frac{r-R}{h} \cdot \frac{h^2}{2} + R^2 h \right]^{x=h}_{x=0} - \left[ \frac{(r-R)^2}{h^2} \cdot \frac{0^3}{3} + \frac{r-R}{h} \cdot \frac{0^2}{2} + R^2 \cdot 0 \right]$$

$$= \pi \frac{(r-R)^2}{h^2} \cdot \frac{h^3}{3} + \frac{r-R}{h} \cdot \frac{h^2}{2} + R^2 h$$

$$= \pi \frac{h(r-R)^2}{3} + Rh(r-R) + R^2 h$$

$$= \pi \frac{h(r-R)^2 + 3Rh(r-R) + 3R^2 h}{3}$$

$$= \pi \frac{h(r^2 - 2rR + R^2) + 3rRh - 3R^2 h + 3R^2 h}{3}$$

$$= \pi \frac{r^2 h - 2rRh + R^2 h + 3rRh}{3}$$

$$= \pi \frac{r^2 h + rRh + R^2 h}{3}$$

Thus, the volume of a frustum with top radius $r$, bottom radius $R$, and height $h$ is given by

$$V_{frustum} = \frac{\pi h}{3} \left( r^2 + rR + R^2 \right).$$