Find the volume of the frustum of a right circular cone with height h, lower base radius R, and top radius r.[1]

A *frustum* of a cone is the part of the cone that remains after the top of the cone is cut-off parallel to the base of the cone. See Figure 1.

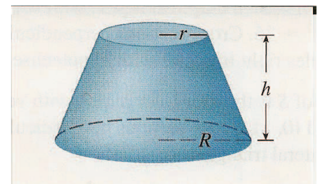

Figure 1

Since the frustum has rotational symmetry, let's set it up so a representative rectangle is rotated about the x-axis.

The slope of the line is $\frac{r-R}{h-0} = \frac{r-R}{h}$, and the y-intercept is R. Using slope-intercept form, the equation of the line that represents the side of the frustum is $y = \frac{r-R}{h}x + R$.

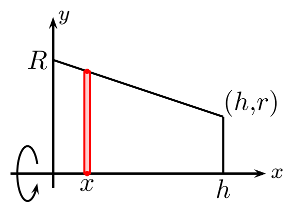

Figure 2

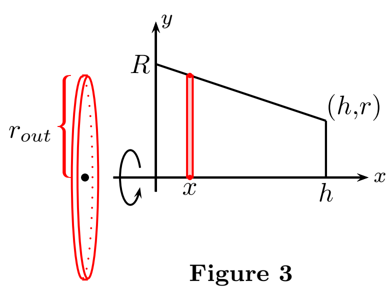

Figure 3

In Figure 3, we've sketched in the disk (washer with no hole) that is generated from revolcing the representative rectangle about the x-axis. We have

$$r_{out} = \frac{r-R}{h}x + R$$
$$r_{in} = 0$$

so the volume of the disk is

$$\begin{aligned} V_{disk} &= \left(\pi\left(\frac{r-R}{h}x + R\right)^2 - \pi\left(0\right)^2\right)\Delta x \\ &= \pi\left(\frac{\left(r-R\right)^2}{h^2}x^2 + 2R\cdot\frac{r-R}{h}x + R^2\right)\Delta x \end{aligned}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 447, #48.

We create disks from $x = 0$ to $x = h$, so the volume of the solid of reveolution, *i.e.*, the volume of the frustum, is given by

$$\begin{aligned} V_{frustum} &= \int_{x=0}^{x=h} \pi \left(\frac{(r-R)^2}{h^2} x^2 + 2R \cdot \frac{r-R}{h} x + R^2 \right) \, \mathrm{d}x \\ &= \pi \int_{x=0}^{x=h} \frac{(r-R)^2}{h^2} x^2 + 2R \cdot \frac{r-R}{h} x + R^2 \, \mathrm{d}x \end{aligned}$$

The integrand looks dreadfully complicated, but keep in mind that all of the coefficients with r, R, and h are constants and the integration is with respect to x. We find an antiderivative for the integrand and apply the Fundamental Theorem of Calculus.

$$\begin{aligned} &= \pi \left[\frac{(r-R)^2}{h^2} \cdot \frac{x^3}{3} + 2R \cdot \frac{r-R}{h} \cdot \frac{x^2}{2} + R^2 x \right]_{x=0}^{x=h} \\ &= \pi \left(\frac{(r-R)^2}{h^2} \cdot \frac{h^3}{3} + 2R \cdot \frac{r-R}{h} \cdot \frac{h^2}{2} + R^2 h \right) - \left(\frac{(r-R)^2}{h^2} \cdot \frac{0^3}{3} + 2R \cdot \frac{r-R}{h} \cdot \frac{0^2}{2} + R^2 \cdot 0 \right) \\ &= \pi \frac{(r-R)^2}{h^2} \cdot \frac{h^3}{3} + 2R \cdot \frac{r-R}{h} \cdot \frac{h^2}{2} + R^2 h \\ &= \pi \frac{h(r-R)^2}{3} + Rh(r-R) + R^2 h \\ &= \pi \frac{h(r-R)^2 + 3Rh(r-R) + 3R^2 h}{3} \\ &= \pi \frac{h\left(r^2 - 2rR + R^2\right) + 3rRh - 3R^2 h + 3R^2 h}{3} \\ &= \pi \frac{r^2 h - 2rRh + R^2 h + 3rRh}{3} \\ &= \pi \frac{r^2 h + rRh + R^2 h}{3} \\ &= \frac{\pi h}{3} \left(r^2 + rR + R^2 \right) \end{aligned}$$

Thus, the volume of a frustum with top radius r, bottom radius R, and height h is given by

$$V_{frustum} = \frac{\pi h}{3} \left(r^2 + rR + R^2 \right).$$