Use the method of cylindrical shells to find the volume of the solid generated by rotating the region bounded by the given curves about the specified axis.[1]

$$x = 2y^2, \quad x = y^2 + 1; \quad \textit{about } x = -2$$

In Figure 1, the curves are sketched along with the axis of revolution. We can find the points of intersection by solving

$$\begin{aligned} y^2 + 1 &= 2y^2 \\ 1 &= y^2 \\ 0 &= y^2 - 1 \\ 0 &= (y+1)(y-1) \end{aligned}$$

So, by the zero product property,

$$y = -1 \quad \text{or} \quad y = 1$$

substitution shows the x-coordinate to be $x = 2$.

Since we are going to use cylindrical shells, the representative rectangle is parallel to the axis of revolution; this is sketched in Figure 2. The width of the representative rectangle is Δy, so everything in our integral will have to be in terms of y. The functions $x = 2y^2$ and $x = y^2 + 1$ are already in terms of y, so we are ready to proceed. In Figure 2, the value of y for the representative rectangle just happens to be negative. The radius of the shell is $r_{\text{shell}} = (\text{larger } y\text{-value}) - (\text{smaller } y\text{-value})$ or $r_{\text{shell}} = y - (-2)$. If we happened to place the representative rectangle with a positive value of y, the radius of the shell is still $r_{\text{shell}} = (\text{larger } y\text{-value}) - (\text{smaller } y\text{-value})$ or $r_{\text{shell}} = y - (-2)$.

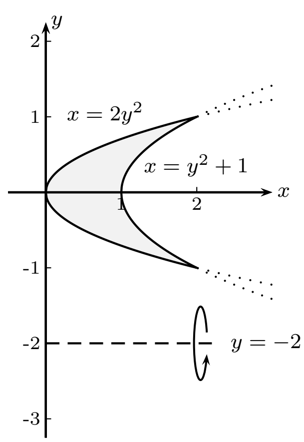

Figure 1: *The curves and the region.*

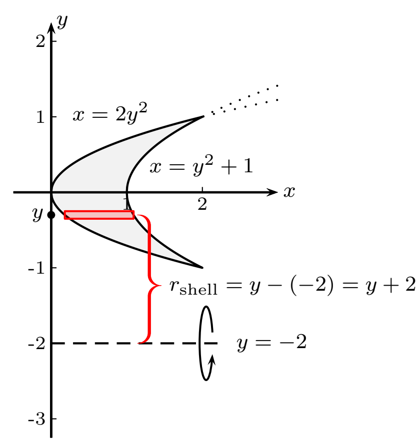

Figure 2: *The region with a representative rectangle.*

In Figure 3, we've sketched the shell generated by revolving the representative rectangle about $y = -2$. The shell, after it has been cut and flattened into a (very skinny) box, is sketched above. The volume of the shell is given by

$$V_{\text{shell}} = (\text{length of box})\,(\text{width of box})\,(\text{thickness of box})$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 454, #20.

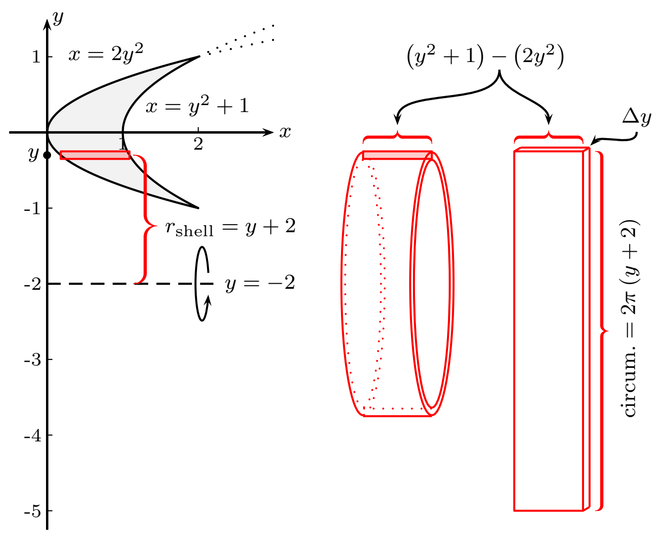

Figure 3: *The shell corresponding to the representative rectangle. The length of the flattened shell is not drawn to scale.*

and since the length of the box is the circumference of the shell

$$= (2\pi\,(y+2))\left(\left(y^2+1\right) - \left(2y^2\right)\right)(\Delta x)$$

We create shells from $y = -1$ to $y = 1$, so the volume, V, of the solid of revolution is given by

$$\begin{aligned}
V &= \int_{y=-1}^{y=1} 2\pi\,(y+2)\left(\left(y^2+1\right) - \left(2y^2\right)\right)\,\mathrm{d}y \\
&= 2\pi \int_{y=-1}^{y=1} (y+2)\left(-y^2+1\right)\,\mathrm{d}y \\
&= 2\pi \int_{y=-1}^{y=1} -y^3 + y - 2y^2 + 2\,\mathrm{d}y \\
&= 2\pi \int_{y=-1}^{y=1} -y^3 - 2y^2 + y + 2\,\mathrm{d}y \\
&= 2\pi \left[-\frac{y^4}{4} - 2\cdot\frac{y^3}{3} - y^2 + 2y\right]_{y=-1}^{y=1} \\
&= 2\pi \left[\left(-\frac{1^4}{4} - \frac{2\cdot 1^3}{3} - 1^2 + 2\cdot 1\right) - \left(-\frac{(-1)^4}{4} - \frac{2\cdot(-1)^3}{3} - (-1)^2 + 2\,(-1)\right)\right] \\
&= 2\pi \left[-\frac{1}{4} - \frac{2}{3} - 1 + 2 + \frac{1}{4} - \frac{2}{3} + 1 + 2\right] \\
&= 2\pi \cdot \frac{8}{3} \\
&= \frac{16}{3}\pi
\end{aligned}$$

Thus the volume of the solid obtained by rotating the region bounded by $x = 2y^2$ and $x = y^2 + 1$ about $y = -2$ is $\frac{16}{3}\pi$ units^3.