If the work required to stretch a spring 1 ft *beyond its natural length is* 12 ft-lb*, how much work is needed to stretch it* 9 in. *beyond its natural length?*[1]

Hooke's Law tells us that the force to stretch a spring x units beyond its natural length is

$$f(x) = kx$$

where k is a positive constant.[2] The phrase "…1 ft beyond its natural length …" tells us x is changing from 0 ft to 1 ft. We also know

$$W = \int_a^b f(x)\ \mathrm{d}x$$

Substituting the given information

$$\begin{aligned} W &= \int_a^b f(x)\ \mathrm{d}x \\ 12 &= \int_0^1 kx\ \mathrm{d}x \end{aligned}$$

and since k is a constant

$$\begin{aligned} 12 &= k \cdot \int_0^1 x\ \mathrm{d}x \\ 12 &= k \cdot \left[\frac{x^2}{2}\right]_0^1 \\ 12 &= k \cdot \left[\frac{1^2}{2} - \frac{0^2}{2}\right] \\ 12 &= k \cdot \left[\frac{1}{2} - 0\right] \\ 12 &= \frac{1}{2}k \\ 24 &= k \end{aligned}$$

So our force function is $f(x) = 24x$, and, for this particular spring, we have $W = \int_a^b 24x\ \mathrm{d}x$.

The problem asks "…how much work is needed to stretch it 9 in. beyond its natural length?" Our integral uses feet as the length unit, so we must recognize 9 in. $= \frac{3}{4}$ ft. The work is given by

$$\begin{aligned} W &= \int_0^{3/4} 24x\ \mathrm{d}x \\ &= \left[24 \cdot \frac{x^2}{2}\right]_{x=0}^{x=3/4} \\ &= \left(24 \cdot \frac{(3/4)^2}{2}\right) - \left(24 \cdot \frac{(0)^2}{2}\right) \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 459, #10.

[2] Note that the statement of Hooke's Law is about *stretching* the spring, so x is a positive number and the force $f(x)$ is a positive number, thus k must be positive. If we were to compress the spring, then x is a negative number, the force $f(x)$ is a negative number, so k remains a positive constant.

$$
\begin{aligned}
&= \left(24 \cdot \frac{9/16}{2}\right) - 0 \\
&= 24 \cdot \frac{9}{32} \\
&= \frac{27}{4}
\end{aligned}
$$

Thus the work required to stretch the spring 9 in. beyond its natural length is $\frac{27}{4} = 6.75$ ft-lb.