A thick cable, 60 ft *long and weighing* 180 lb*, hangs from a winch on a crane. Compute in two different ways the work done if the winch winds up* 25 ft *of cable.*[1]

Recall that for a *constant force*,

work = force · distance

There are two ways to view this product of force and distance in work problems[2]

1. Move a small part of the object the entire distance.
2. Move the entire object a small part of the distance.

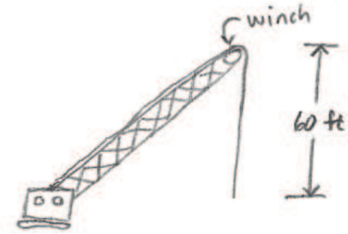

Figure 1: *A cable hangs from a winch on a crane.*

We will examine the product of force and distance over a small part (Δ*something*), and then add all of those products together in a Riemann sum. Computing the limit as the number of small parts tends to infinity[3] will give us a definite integral

For this particular problem, the units are given in terms of feet and pounds. In the U.S. customary system[4], the *pound* is a unit of *force*, and the foot is a unit of distance. What ever integral we write for a work problem must reflect the product of force and distance and careful examination of the units can help insure that we have the correct integral.

1. Move a small part of the object the entire distance.

   Here, the cable hangs from the winch, so the top 25 ft will be wound onto the winch. Since each small part of the top 25 ft will travel a different distance[5], we'll analyze the top 25 ft separately from the bottom 35 ft.

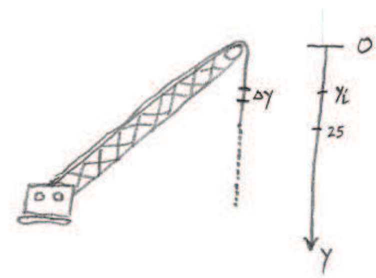

Figure 2: *Moving a small length of cable the entire distance.*

   (a) For the top 25 ft...

   Consider a small length Δy, in feet, of the cable, y_i feet from the winch. Since the cable weighs[6] $\frac{180\,\text{lb}}{60\,\text{ft}} = 3\,\frac{\text{lb}}{\text{ft}}$, our small length weighs

$$(\Delta y\,\text{ft})\left(3\frac{\text{lb}}{\text{ft}}\right) = 3\Delta y\,\text{lb}$$

   This is the <u>force</u> on our small length. This small length moves y_i ft, so

$$W_{\text{small length}} \approx 3\Delta y \cdot y_i$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 459, #14.
[2]Most other applications of variable force, density, *etc.*, can be characterized in the same manner.
[3]Conversely, we can compute the limit as the Δ*something* approaches zero.
[4]Sometimes called the FPS (foot-pound-second) system, the Stroud system, or the Imperial system of measurement.
[5]Those small parts near the winch will travel a short distance, while those farther from the winch will travel a longer distance.
[6]"Weight" in the U.S. customary system is a force.

$$= 3y_i \Delta y$$

We add all these parts together to get the Riemann sum

$$W_{\text{top 25 ft}} \approx \sum_{i=1}^{n} 3y_i \Delta y$$

and we take the limit as the number of parts approaches infinity

$$W_{\text{top 25 ft}} = \lim_{n\to\infty} \sum_{i=1}^{n} 3y_i \Delta y$$

Since we have small lengths from $y = 0\,\text{ft}$ to $y = 25\,\text{ft}$, the definite integral for the work, W_1, to wind the top 25 ft onto the winch is

$$\begin{aligned} W_1 &= \int_{y=0}^{y=25} 3y \; \mathrm{d}y \\ &= \left[\frac{3y^2}{2}\right]_{y=0}^{y=25} \\ &= \left[\frac{3\cdot 25^2}{2} - \frac{3\cdot 0^2}{2}\right] \\ &= \frac{1875}{2}\,\text{ft-lb} \end{aligned}$$

In our integral, $\int_{y=0}^{y=25} 3y \; \mathrm{d}y$ the force is $3 \; \mathrm{d}y$ and the distance is y.

(b) For the lower 35 ft...

The remaining 35 ft will be pulled up the 25 ft as the winch winds up the cable. For the lower 35 ft, each small length Δy moves the same 25 ft, *ie*, the displacement is a constant 25 ft. The force on each small length is $3\Delta y$ lb and each small length moves 25 ft, so the work done on each small length is

$$\begin{aligned} W_{\text{small length}} &\approx 3\Delta y \cdot 25 \\ &\approx 75\Delta y \end{aligned}$$

We add all these parts together to get the Riemann sum

$$W_{\text{lower 35 ft}} \approx \sum_{i=1}^{n} 75\Delta y$$

and we take the limit as the number of parts approaches infinity

$$W_{\text{lower 35 ft}} = \lim_{n\to\infty} \sum_{i=1}^{n} 75\Delta y$$

Since we have small lengths from $y = 25\,\text{ft}$ to $y = 60\,\text{ft}$, the definite integral for the work, W_2, to lift the lower 55 ft is

$$\begin{aligned} W_2 &= \int_{y=25}^{y=60} 75 \; \mathrm{d}y \\ &= [75y]_{y=25}^{y=60} \\ &= 75\cdot 60 - 75\cdot 25 \\ &= 2625\,\text{ft-lb} \end{aligned}$$

Thus, the total work is $W_1 + W_2 = 937.5\,\text{ft-lb} + 2625\,\text{ft-lb} = 3562.5\,\text{ft-lb}$.

2. Move the entire object a small part of the distance.

Let's write an expression for the weight (force) on the cable that is hanging down from the winch after y_i feet of the cable has been pulled up onto the winch. If y_i feet have been pulled up, then $60 - y_i$ feet are still hanging. That part of the cable weighs

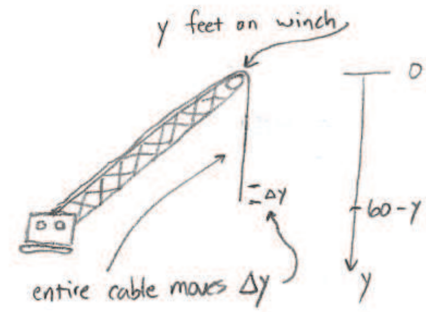

Figure 3: *Moving the entire cable a small distance.*

$$(60\,\text{ft} - y_i\,\text{ft})\left(3\frac{\text{lb}}{\text{ft}}\right) = 3\,(60 - y_i)\ \text{lb}$$

We move the entire hanging portion of the cable Δy feet. So the work done moving the entire cable a small part of the distance is

$$W_{\text{small distance}} \approx 3\,(60 - y_i)\,\Delta y$$

We add all these parts together to get the Riemann sum

$$W_{\text{small distance}} \approx \sum_{i=1}^{n} 3\,(60 - y_i)\,\Delta y$$

and we take the limit as the number of small distances approaches infinity[7]

$$W_{\text{small distance}} = \lim_{n\to\infty} \sum_{i=1}^{n} 3\,(60 - y_i)\,\Delta y$$

Since our distance change from $y = 0$ to $y = 25$, we get the definite integral

$$\begin{aligned} W &= \int_{y=0}^{y=25} 3\,(60 - y)\ \mathrm{d}y \\ &= 3 \cdot \int_{y=0}^{y=25} 60 - y\ \mathrm{d}y \\ &= 3 \cdot \left[60y - \frac{y^2}{2}\right]_{y=0}^{y=25} \\ &= 3 \cdot \left[\left(60 \cdot 25 - \frac{25^2}{2}\right) - \left(60 \cdot 0 - \frac{0^2}{2}\right)\right] \\ &= 3 \cdot \left[1500 - \frac{625}{2} - 0 + 0\right] \\ &= 3562.5 \end{aligned}$$

Thus, the work is $W = 3562.5\,\text{ft-lb}$.

[7] As the number of small distance approaches infinity, the length of each small distance approaches zero.