A bucket that weighs 4 lb *and a rope of negligible weight are used to draw water from a well that is* 80 ft *deep. The bucket is filled with* 40 lb *of water and is pulled up at the rate of* 2 ft/s, *but water leaks out of a hole in the bucket at the rate of* 0.2 lb/s. *Find the work done in pulling the bucket to the top of the well.*[1]

For this situation, there are two things being pulled to the top of the well.

1. The bucket.
2. The water.

We will find the work on each of these separately, and then take the sum as the total work.

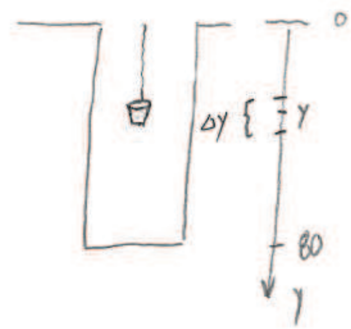

Figure 1: *A bucket being pulled from a well.*

1. For the bucket...

   The weight (force) of the bucket is a constant 4 lb, and we'll move the bucket a small distance Δy. We have these small distances from $y = 0$ to $y = 80$, so

$$\begin{aligned} W_{\text{bucket}} &= \int_{y=0}^{y=80} 4 \, \mathrm{d}y \\ &= [4y]_{y=0}^{y=80} \\ &= 4 \cdot 80 - 4 \cdot 0 \\ &= 320 \end{aligned}$$

   Thus the work to lift the bucket out of the well is 320 ft-lb.

2. For the water...

   Water leaks out of the bucket at a constant rate of 0.2 lb/s, so the weight (force) of the water can be modeled with a linear function. The input for this function will be the value y, in feet, and the output will be the weight (force) of the water at that depth in the well.

   Since the well is 80 ft deep, it takes

$$(80\,\text{ft})\left(\frac{1\,\text{s}}{2\,\text{ft}}\right) = 40\,\text{s}$$

   to pull the bucket to the top of the well. After those 40 s, we lost

$$\left(\frac{0.2\,\text{lb}}{1\,\text{s}}\right)(40\,\text{s}) = 8\,\text{lb}$$

   of water, so 32 lb of water remains. From Figure 1, when $y = 0$, weight (force) $= 32$ and when $y = 80$, weight (force) $= 40$. The slope of the function is

$$\begin{aligned} \text{slope} &= \frac{40 - 32}{80 - 0} \\ &= \frac{1}{10} \end{aligned}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 459, #18.

We can start building our linear function

$$\text{weight} = \frac{1}{10}y + b$$

substitute

$$\begin{aligned} 32 &= \frac{1}{10} \cdot 0 + b \\ b &= 32 \end{aligned}$$

So the function giving the weight (force) of the water at any depth y in the well is weight $= \frac{1}{10}y + 32$.

If we move a small distance Δy, and we have these distances from $y = 0$ to $y = 80$, we get

$$\begin{aligned} W_{\text{water}} &= \int_{y=0}^{y=80} \frac{1}{10}y + 32 \, \mathrm{d}y \\ &= \left[\frac{1}{10} \cdot \frac{y^2}{2} + 32y\right]_{y=0}^{y=80} \\ &= \left[\left(\frac{80^2}{20} + 32 \cdot 80\right) - \left(\frac{0^2}{20} + 32 \cdot 0\right)\right] \\ &= 2880 \end{aligned}$$

Thus the total work, W, is

$$W = W_{\text{bucket}} + W_{\text{water}} = 320\,\text{ft-lb} + 2880\,\text{ft-lb} = 3200\,\text{ft-lb}$$