A tank is full of water. Find the work required to pump the water out of the spout.[1]

As we have done many times, and as we will continue to do with virtually all applications, we will analyze what happens to one small part of the object, write the Riemann sum for all the small parts, and finally add up all the small parts with integration.

For the tank in Figure 1, we will find an expression for the length and width at any depth y in the tank, find the volume of a representative slice at that depth, find the force (weight) on the slice, Find how far each representative slice moves, and finally form the integral for the work.

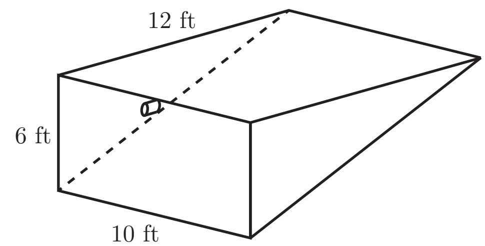

Figure 1: *A tank filled with water.*

In Figure 2, we've sketched a representative slice. As usual, the thickness of the representative slice is Δy. As the depth y varies, the width of the slice and the thickness of the slice are constant at 6 ft and Δy, respectively. The length of the slice (parallel to the 12 ft edge of the tank) is changing. To compute the volume of the slice, we must express this length in terms of

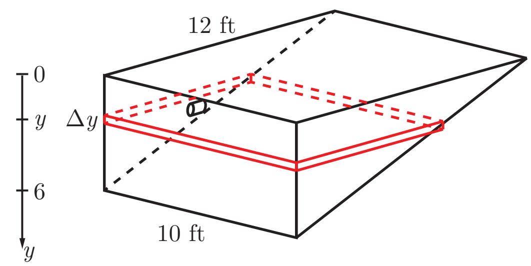

Figure 2: *A representative slice.*

In Figure 3, we've sketched a cross-section of the side of the tank, along with the length, l, of the representative slice. To find l in terms of y, we use similar triangles.

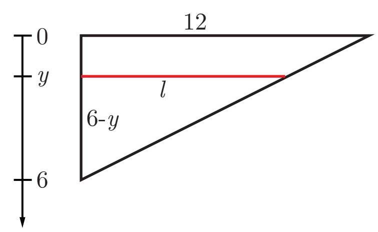

Figure 3: *Cross-section of the tank.*

$$\begin{aligned} \frac{l}{6-y} &= \frac{12}{6} \\ 6(6-y)\frac{l}{6-y} &= 6(6-y)\frac{12}{6} \\ 6l &= 12(6-y) \\ l &= \frac{1}{6}(72-12y) \\ l &= 12-2y \end{aligned}$$

We compute

$$\begin{aligned} V_{\text{slice}} &= (12-2y)(10)\Delta y \\ &= 20(6-y)\Delta y \end{aligned}$$

For this problem we are using the U.S. customary system of ft-lb-s. We are told that water weighs 62.5 lb/ft^3, so the weight (force) of the slice is given by

$$\text{Force}_{\text{slice}} = 62.5 \cdot 20(6-y)\Delta y$$

and each slice moves y ft, so

$$\text{Work}_{\text{slice}} = 62.5 \cdot 20(6-y)\Delta y \cdot y$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 460, #26.

$$= 1250y\,(6-y)\,\Delta y$$

We create slices from $y = 0$ to $y = 6$, so the work is given by

$$\begin{aligned}
W &= \int_{y=0}^{y=6} 1250y\,(6-y)\ \mathrm{d}y \\
&= 1250\int_{y=0}^{y=6} y\,(6-y)\ \mathrm{d}y \\
&= 1250\int_{y=0}^{y=6} 6y - y^2\ \mathrm{d}y \\
&= 1250\left[3y^2 - \frac{y^3}{3}\right]_{y=0}^{y=6} \\
&= 1250\left[\left(3\cdot 6^2 - \frac{6^3}{3}\right) - \left(3\cdot 0^2 - \frac{0^3}{3}\right)\right] \\
&= 1250\,[108 - 72 - 0 + 0] \\
&= 45{,}000\,\text{ft-lb}
\end{aligned}$$

Thus, the amount of work to pump all the water out of this tank is 45,000 ft-lb.