Evaluate the integral.[1]

$$\int \frac{\mathrm{d}t}{t^2\sqrt{t^2-16}}$$

$$\int \frac{\mathrm{d}t}{t^2\sqrt{t^2-16}} = \int \frac{1}{t^2\sqrt{t^2-4^2}}\,\mathrm{d}t$$

The radical has the form $\sqrt{x^2-a^2}$ so we make the subsitution

$$\begin{aligned} \text{Let} \quad t &= 4\sec(\theta) \\ \text{so} \quad \mathrm{d}t &= 4\sec(\theta)\tan(\theta)\,\mathrm{d}\theta \end{aligned}$$

Then we have

$$t^2 = 16\sec^2(\theta)$$

and

$$\begin{aligned} t^2-16 &= 16\sec^2(\theta)-16 \\ &= 16\left(\sec^2(\theta)-1\right) \end{aligned}$$

Now $1+\tan^2(\theta)=\sec^2(\theta)$, so $\sec^2(\theta)-1=\tan^2(\theta)$, and we have

$$t^2-16 = 16\tan^2(\theta)$$

so

$$\begin{aligned} \sqrt{t^2-16} &= \sqrt{16\tan^2(\theta)} \\ &= 4\sqrt{\tan^2(\theta)} \\ &= 4\left|\tan(\theta)\right| \\ &= 4\tan(\theta), \quad \theta \in \text{QI or III.} \end{aligned}$$

Since $\sqrt{a^2}=|a|$.

For $|\tan(\theta)|$ to equal $\tan(\theta)$, we must have $\tan(\theta)$ nonnegative, which means we must have θ in Quadrant I or III.

We substitute all these pieces in terms of θ into the integral

$$\begin{aligned} \int \frac{1}{t^2\sqrt{t^2-16}}\,\mathrm{d}t &= \int \frac{1}{16\sec^2(\theta)\cdot 4\tan(\theta)} \cdot 4\sec(\theta)\tan(\theta)\ \mathrm{d}\theta \\ &= \int \frac{1}{16\sec(\theta)}\,\mathrm{d}\theta \\ &= \frac{1}{16}\cdot\int \frac{1}{\sec(\theta)}\,\mathrm{d}\theta \\ &= \frac{1}{16}\cdot\int \cos(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{16}\cdot\left(\sin(\theta)+C\right) \\ &= \frac{1}{16}\sin(\theta)+\frac{1}{16}C \end{aligned}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 491, #8.

Since C is an arbitrary constant, $\frac{1}{16}C$ is also an arbitrary constant, so we'll just write

$$= \frac{1}{16}\sin(\theta) + C$$

We have succeeded in integrating our function, but the result is in terms of θ rather than t, so we'll try to substitute back to get the result in terms of t. If we look at our initial substitution, $t = 4\sec(\theta)$, and other calculations, $t^2 - 16 = 16\tan^2(\theta)$ or $\sqrt{t^2-16} = 4\tan(\theta)$, none of them involve the $\sin(\theta)$ that we got from the integration. In this case, we use $t = 4\sec(\theta)$ and a right triangle to express $\sin(\theta)$ in terms of t.

Since $t = 4\sec(\theta)$, we have

$$t = 4 \cdot \frac{1}{\cos(\theta)}$$

or

$$\cos(\theta) = \frac{4}{t}$$

and we use this relationship to sketch the triangle showun in Figure 1. (We could've used $\frac{t}{4} = \sec(\theta)$, but most of us are more comfortable with the cosine function.)

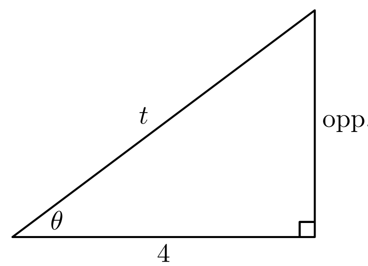

Figure 1

From Pythagorean Theorem, we get

$$\begin{aligned} 4^2 + \text{opp.}^2 &= t^2 \\ 16 + \text{opp.}^2 &= t^2 \\ \text{opp.}^2 &= t^2 - 16 \end{aligned}$$

And since the opposite side is a length and thus non-negative,

$$\text{opp.} = \sqrt{t^2-16}$$

In Figure 2, the three sides of the triangle are labeled. From the figure, we see

$$\sin(\theta) = \frac{\sqrt{t^2-16}}{t}$$

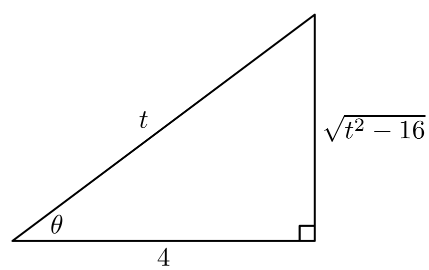

Figure 2

Finally, substituting gives us

$$\begin{aligned} \int \frac{1}{t^2\sqrt{t^2-16}}\,\mathrm{d}t &= \frac{1}{16}\sin(\theta) + C \\ &= \frac{1}{16} \cdot \frac{\sqrt{t^2-16}}{t} + C \\ &= \frac{\sqrt{t^2-16}}{16t} + C \end{aligned}$$

Thus,

$$\int \frac{1}{t^2\sqrt{t^2-16}}\,\mathrm{d}t = \frac{\sqrt{t^2-16}}{16t} + C$$