Evaluate the integral.[1]

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

With $\sqrt{1+x^2}$ in the integrand, it seems like this integral is ready for a trig substitution.

The radical has the form $\sqrt{a^2+x^2}$ so we make the subsitution

$$\begin{aligned} \text{Let} \quad x &= \tan(\theta) \\ \text{so} \quad \mathrm{d}x &= \sec^2(\theta)\,\mathrm{d}\theta \end{aligned}$$

Then we have

$$x^2 = \tan^2(\theta)$$

and

$$\begin{aligned} 1+x^2 &= 1+\tan^2(\theta) \\ &= \sec^2(\theta) \end{aligned}$$

so

$$\begin{aligned} \sqrt{1+x^2} &= \sqrt{\sec^2(\theta)} \\ &= |\sec(\theta)| \\ &= \sec(\theta)\,, \quad \theta \in \text{QI or IV}. \end{aligned}$$

We substitute all these pieces in terms of θ into the integral

$$\begin{aligned} \int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x &= \int \frac{\tan(\theta)}{\sec(\theta)} \cdot \sec^2(\theta)\ \mathrm{d}\theta \\ &= \int \tan(\theta)\sec(\theta)\ \mathrm{d}\theta \\ &= \sec(\theta) + C \end{aligned}$$

We have succeeded in integrating our function, but the result is in terms of θ rather than x, so we'll substitute to get the result in terms of x. If we look at our initial substitution, $x = \tan(\theta)$, and other calculations, $1+x^2 = \sec^2(\theta)$ or $\sqrt{1+x^2} = \sec(\theta)$, this last equation gives us what we need.[2]

Thus,

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = \sqrt{1+x^2} + C$$

OR. . .

[1] Stewart, *Calculus, Early Transcendentals*, p. 491, #20.

[2] We could also draw a right triangle and use $x = \tan(\theta)$ to find the sides in terms of x, but we don't need to do so in this problem.

If we look at the integrand carefully,

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

we see that the derivative of $1+x^2$ is (almost) in the numerator. Let's try a u-substitution.

Let $u = 1+x^2$, so $\mathrm{d}u = 2x\mathrm{d}x$. Then

$$\begin{aligned}
&= \frac{1}{2}\cdot\int \frac{2x}{\sqrt{1+x^2}}\,\mathrm{d}x \\
&= \frac{1}{2}\cdot\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{2}\cdot\int u^{-1/2}\,\mathrm{d}u \\
&= \frac{1}{2}\cdot\left[\frac{u^{-1/2+1}}{-1/2+1}+C\right] \\
&= \frac{1}{2}\cdot\left[\frac{u^{1/2}}{1/2}+C\right] \\
&= \frac{1}{2}\cdot\left[2u^{1/2}+C\right] \\
&= \frac{1}{2}\cdot 2u^{1/2}+\frac{1}{2}\cdot C
\end{aligned}$$

Since C is an arbitrary constant, $\frac{1}{2}\cdot C$ is also an arbitrary constant, and we just write

$$\begin{aligned}
&= \frac{1}{2}\cdot 2u^{1/2}+C \\
&= u^{1/2}+C \\
&= \left(1+x^2\right)^{1/2}+C \\
&= \sqrt{1+x^2}+C
\end{aligned}$$

Thus,

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = \sqrt{1+x^2}+C$$