Evaluate the integral.[1]

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,\mathrm{d}x$$

Does the integrand match one of our basic indefinite integral patterns? No. Can we do a basic u-substitution? No. How about integration by parts? No. Is the integrand powers of trig functions? Nope. Does the integrand include a square root that matches one of our three trig substitutions? No... hold on. There *is* a square root in the denominator:

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,\mathrm{d}x = \int \frac{x^2}{\left(\sqrt{3+4x-4x^2}\,\right)^3}\,\mathrm{d}x$$

Maybe we can complete the square of the radicand to match one of our trigonometric substitutions...

$$\begin{aligned} 3+4x-4x^2 &= 3+\left(4x-4x^2\right) \\ &= 3+(-1)\left(4x^2-4x\right) \\ &= 3+(-1)(4)\left(x^2-x\right) \end{aligned}$$

We compute one-half the linear coefficient, $\frac{1}{2}\cdot -1 = -\frac{1}{2}$, square it, $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$, and add it to our quadratic expression

$$= 3-(4)\left(x^2-x+\frac{1}{4}\right)+4\cdot\frac{1}{4}$$

When we added $\frac{1}{4}$ in the parentheses, we really added $-4\cdot\frac{1}{4} = -1$ to the expression, so we must add $4\cdot\frac{1}{4} = 1$ so we do not change the value of the expression.

$$\begin{aligned} &= 3-(4)\left(x^2-x+\frac{1}{4}\right)+1 \\ &= 4-4\left(x^2-x+\frac{1}{4}\right) \\ &= 4-4\left(x-\frac{1}{2}\right)^2 \\ &= 4-2^2\left(x-\frac{1}{2}\right)^2 \\ &= 4-\left(2\cdot\left(x-\frac{1}{2}\right)\right)^2 \\ &= 2^2-(2x-1)^2 \end{aligned}$$

So our integral becomes

$$= \int \frac{x^2}{\left(\sqrt{2^2-(2x-1)^2}\right)^3}\,\mathrm{d}x$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 491, #26.

The radicand now has the form $a^2 - x^2$, so we let $2x - 1 = 2\sin(\theta)$, and $2\mathrm{d}x = 2\cos(\theta)\,\mathrm{d}\theta$. Also

$$\begin{aligned}(2x-1)^2 &= (2\sin(\theta))^2 \\ (2x-1)^2 &= 4\sin^2(\theta) \\ 2^2 - (2x-1)^2 &= 2^2 - 4\sin^2(\theta) \\ &= 4\left(1-\sin^2(\theta)\right) \\ &= 4\cos^2(\theta)\end{aligned}$$

So

$$\begin{aligned}\sqrt{2^2-(2x-1)^2} &= \sqrt{4\cos^2(\theta)} \\ &= 2\left|\cos(\theta)\right| \\ &= 2\cos(\theta)\,, \quad \theta \in \text{QI, QIV}\end{aligned}$$

Finally, since

$$\begin{aligned}2x-1 &= 2\sin(\theta) \\ 2x &= 2\sin(\theta)+1 \\ x &= \frac{1}{2}(2\sin(\theta)+1) \\ x^2 &= \frac{1}{4}(2\sin(\theta)+1)^2\end{aligned}$$

Now we can substitute into our integral

$$\begin{aligned}\int \frac{x^2}{\left(\sqrt{3+4x-4x^2}\,\right)^3}\,\mathrm{d}x &= \int \frac{\frac{1}{4}(2\sin(\theta)+1)^2}{(2\cos(\theta))^3}\cos(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{4}\int \frac{(2\sin(\theta)+1)^2}{2^3\cos^3(\theta)}\cos(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{32}\int \frac{4\sin^2(\theta)+4\sin(\theta)+1}{\cos^2(\theta)}\ \mathrm{d}\theta \\ &= \frac{1}{32}\int \frac{4\sin^2(\theta)}{\cos^2(\theta)} + \frac{4\sin(\theta)}{\cos^2(\theta)} + \frac{1}{\cos^2(\theta)}\ \mathrm{d}\theta \\ &= \frac{1}{32}\int 4\tan^2(\theta) + 4\tan(\theta)\sec(\theta) + \sec^2(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{32}\int 4\left(\sec^2(\theta)-1\right) + 4\tan(\theta)\sec(\theta) + \sec^2(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{32}\int 5\sec^2(\theta) - 4 + 4\tan(\theta)\sec(\theta)\ \mathrm{d}\theta \\ &= \frac{1}{32}\left[5\tan(\theta) - 4\theta + 4\sec(\theta)\right] + C \\ &= \frac{5}{32}\tan(\theta) - \frac{1}{8}\theta + \frac{1}{8}\sec(\theta) + C\end{aligned}$$

None of our earlier substitutions involved $\tan(\theta)$ or $\sec(\theta)$, so we'll draw a right triangle to find those functions in terms of x. Since $2x-1 = 2\sin(\theta)$ and $\sin(\theta) = \frac{2x-1}{2}$, we get the triangle shown in Figure 1—after a little Pythagorean Theorem.

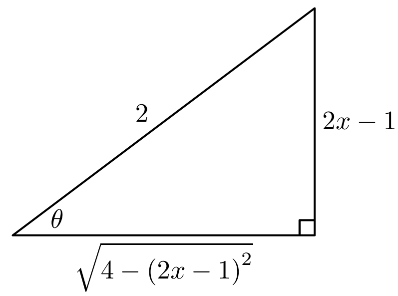

Figure 1

From the triangle we get

$$\tan(\theta) = \frac{2x-1}{\sqrt{4-(2x-1)^2}}, \qquad \sec(\theta) = \frac{2}{\sqrt{4-(2x-1)^2}}, \qquad \theta = \sin^{-1}\left(\frac{2x-1}{2}\right)$$

and after substituting, we have

$$\begin{aligned}
&= \frac{5}{32}\frac{2x-1}{\sqrt{4-(2x-1)^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + \frac{1}{8}\frac{2}{\sqrt{4-(2x-1)^2}} + C \\
&= \frac{5(2x-1)}{32\sqrt{4-(2x-1)^2}} + \frac{8}{32\sqrt{4-(2x-1)^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C \\
&= \frac{10x+3}{32\sqrt{4-(2x-1)^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C \\
&= \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C
\end{aligned}$$

Thus,

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,\mathrm{d}x = \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$