Calculus II, Section 7.3, #42
Trigonometric Integrals

A charged rod of length L produces an electric field at point $P\,(a,b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0 \left(x^2+b^2\right)^{3/2}}\,\mathrm{d}x$$

where λ is the charge density per unit length on the rod and $\epsilon_)$ is the free space permittivity (see Figure 1). Evaluate the integral to determine an expression for the electric field $E(P)$.[1]

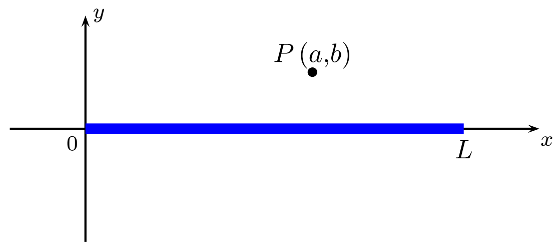


Figure 1: *A charged rod.*

We are integrating with respect to x, so the other parameters–λ, ϵ_0, and b–are constants, so we can factor them out of the integrand.

$$\begin{aligned}
E(P) &= \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0 \left(x^2+b^2\right)^{3/2}}\,\mathrm{d}x \\
&= \frac{\lambda b}{4\pi\epsilon_0} \int_{-a}^{L-a} \frac{1}{\left(x^2+b^2\right)^{3/2}}\,\mathrm{d}x \\
&= \frac{\lambda b}{4\pi\epsilon_0} \int_{-a}^{L-a} \frac{1}{\left(\sqrt{x^2+b^2}\right)^{3}}\,\mathrm{d}x
\end{aligned}$$

The radicand is of the form b^2+x^2, so we let $x = b\tan(\theta)$ and $\mathrm{d}x = b\sec^2(\theta)\,\mathrm{d}\theta$.

$$\begin{aligned}
\sqrt{x^2+b^2} &= \sqrt{b^2\tan^2(\theta)+b^2} \\
&= \sqrt{b^2\left(\tan^2(\theta)+1\right)} \\
&= b\sqrt{\sec^2(\theta)} \\
&= b\left|\sec(\theta)\right| \\
&= b\sec(\theta), \quad \theta \in \text{QI or IV}
\end{aligned}$$

When $x=-a$, $-a = b\tan(\theta)$, so $\theta = \tan^{-1}\left(\frac{-a}{b}\right)$. Also, when $x = L-a$, $L-a = b\tan(\theta)$, so $\theta = \tan^{-1}\left(\frac{L-a}{b}\right)$. We get

$$\begin{aligned}
E(P) &= \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0 \left(x^2+b^2\right)^{3/2}}\,\mathrm{d}x \\
&= \frac{\lambda b}{4\pi\epsilon_0} \int_{-a}^{L-a} \frac{1}{\left(\sqrt{x^2+b^2}\right)^{3}}\,\mathrm{d}x \\
&= \frac{\lambda b}{4\pi\epsilon_0} \int_{\theta=\tan^{-1}\left(\frac{-a}{b}\right)}^{\theta=\tan^{-1}\left(\frac{L-a}{b}\right)} \frac{1}{b^3\sec^3(\theta)} \cdot b\sec^2(\theta)\ \mathrm{d}\theta
\end{aligned}$$

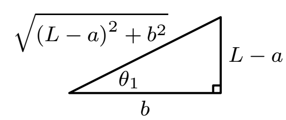
[1] Stewart, *Calculus, Early Transcendentals*, p. 492, #42.

$$= \frac{\lambda}{4\pi b\epsilon_0} \int_{\theta=\tan^{-1}\left(\frac{-a}{b}\right)}^{\theta=\tan^{-1}\left(\frac{L-a}{b}\right)} \frac{1}{\sec(\theta)}\, \mathrm{d}\theta$$
$$= \frac{\lambda}{4\pi b\epsilon_0} \int_{\theta=\tan^{-1}\left(\frac{-a}{b}\right)}^{\theta=\tan^{-1}\left(\frac{L-a}{b}\right)} \cos(\theta)\, \mathrm{d}\theta$$
$$= \frac{\lambda}{4\pi b\epsilon_0} \left[\sin(\theta)\right]_{\theta=\tan^{-1}\left(\frac{-a}{b}\right)}^{\theta=\tan^{-1}\left(\frac{L-a}{b}\right)}$$
$$= \frac{\lambda}{4\pi b\epsilon_0} \left[\sin\left(\tan^{-1}\left(\frac{L-a}{b}\right)\right) - \sin\left(\tan^{-1}\left(\frac{-a}{b}\right)\right)\right]$$

To evaluate $\sin\left(\tan^{-1}\left(\frac{L-a}{b}\right)\right)$, we let

$$\theta_1 = \tan^{-1}\left(\frac{L-a}{b}\right)$$
$$\tan(\theta_1) = \frac{L-a}{b}$$

and we draw the triangle at right, using Pythagorean Theorm to find the hypotenuse. So

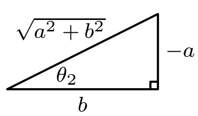


$$\sin(\theta_1) = \frac{L-a}{\sqrt{(L-a)^2 + b^2}}$$

Similarly, to evaluate $\sin\left(\tan^{-1}\left(\frac{-a}{b}\right)\right)$, we let

$$\theta_2 = \tan^{-1}\left(\frac{-a}{b}\right)$$
$$\tan(\theta_2) = \frac{-a}{b}$$

and we draw the triangle at right, using Pythagorean Theorm to find the hypotenuse. So



$$\sin(\theta_2) = \frac{-a}{\sqrt{a^2 + b^2}}$$

Substituting these results back into our last expression for the integral, we get

$$= \frac{\lambda}{4\pi b\epsilon_0} \left[\sin\left(\tan^{-1}\left(\frac{L-a}{b}\right)\right) - \sin\left(\tan^{-1}\left(\frac{-a}{b}\right)\right)\right]$$
$$= \frac{\lambda}{4\pi b\epsilon_0} \left[\frac{L-a}{\sqrt{(L-a)^2 + b^2}} - \frac{-a}{\sqrt{a)^2 + b^2}}\right]$$
$$= \frac{\lambda}{4\pi b\epsilon_0} \left[\frac{L-a}{\sqrt{(L-a)^2 + b^2}} + \frac{a}{\sqrt{a)^2 + b^2}}\right]$$

Thus,

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0 \left(x^2 + b^2\right)^{3/2}}\, \mathrm{d}x = \frac{\lambda}{4\pi b\epsilon_0} \left[\frac{L-a}{\sqrt{(L-a)^2 + b^2}} + \frac{a}{\sqrt{a)^2 + b^2}}\right]$$