Evaluate the integral.[1]

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x$$

Consider the integral

$$\int_{x=0}^{x=1} \frac{x-4}{x^2-5x+6}\,\mathrm{d}x$$

Is the integrand one of our basic indefinite integrals? No. How about a basic u-substitution? No. Integration by parts? No. Powers of trig functions? No. Does the integrand include a trig. sub. radical? No. Is the integrand a rational function? YES!

The degree of the numerator is 1, and the degree of the denominator is 2. Since the degree of the numerator is less than the degree of the denominator, we are ready to begin the partial fractions process.

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-3)(x-2)}$$

$x-3$ is a distinct linear factor, as is $x-2$. We have

$$\frac{x-4}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}$$

The LCD is $(x-3)(x-2)$, and we multiply both sides of this identity by the LCD to get

$$\begin{aligned}
\frac{x-4}{x^2-5x+6}\cdot(x-3)(x-2) &= \frac{A}{x-3}\cdot(x-3)(x-2) + \frac{B}{x-2}\cdot(x-3)(x-2)\\
x-4 &= A(x-2) + B(x-3)\\
x-4 &= Ax - 2A + Bx - 3B\\
x-4 &= Ax + Bx - 2A - 3B\\
x-4 &= (A+B)x + (-2A-3B)
\end{aligned}$$

Since this equation is an identity, *i.e.*, it is true for all allowable values of x, we equate coefficients to get the system of equations

$$\begin{cases} 1 = A + B \\ -4 = -2A - 3B \end{cases}$$

If we multiply 2 times the top equation and add to the bottom equation we get

$$\begin{aligned}
-2 &= -B\\
B &= 2
\end{aligned}$$

and substituting into the top equation gives us

$$A = -1$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 501, #12.

The partial fraction decomposition is

$$\frac{x-4}{x^2-5x+6} = \frac{-1}{x-3} + \frac{2}{x-2}$$

so our integral becomes

$$\begin{aligned}
\int_{x=0}^{x=1} \frac{x-4}{x^2-5x+6}\,\mathrm{d}x &= \int_{x=0}^{x=1} \frac{-1}{x-3} + \frac{2}{x-2}\,\mathrm{d}x \\
&= \int_{x=0}^{x=1} \frac{-1}{x-3}\,\mathrm{d}x + \int_{x=0}^{x=1} \frac{2}{x-2}\,\mathrm{d}x \\
&= -1\int_{x=0}^{x=1} \frac{1}{x-3} + 2\int_{x=0}^{x=1} \frac{1}{x-2}\,\mathrm{d}x \\
&= -1\left[\ln|x-3|\right]_{x=0}^{x=1} + 2\left[\ln|x-2|\right]_{x=0}^{x=1} \\
&= -1\left[\ln|1-3| - \ln|0-3|\right] + 2\left[\ln|1-2| - \ln|0-2|\right] \\
&= -1\left[\ln(2) - \ln(3)\right] + 2\left[\ln(1) - \ln(2)\right] \\
&= -\ln(2) + \ln(3) + 2\ln(1) - 2\ln(2) \\
&= -3\ln(2) + \ln(3) \\
&= \ln\left(2^{-3}\right) + \ln(3) \\
&= \ln\left(\frac{1}{2^3}\right) + \ln(3) \\
&= \ln\left(\frac{1}{8}\right) + \ln(3) \\
&= \ln\left(\frac{3}{8}\right)
\end{aligned}$$

Thus,

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x = \ln\left(\frac{3}{8}\right)$$