Evaluate the integral.[1]

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x$$

Consider the integral

$$\int_{x=2}^{x=3} \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x$$

Is the integrand one of our basic indefinite integrals? No. How about a basic u-substitution? No. Integration by parts? No. Powers of trig functions? No. Does the integrand include a trig. sub. radical? No. Is the integrand a rational function? YES!

The degree of the numerator is 2, and the degree of the denominator is 3. Since the degree of the numerator is less than the degree of the denominator, we are ready to begin the partial fractions process.

Consider the integrand

$$\frac{x(3-5x)}{(3x-1)(x-1)^2}$$

$3x-1$ is a distinct linear factor, whereas $x-1$ is a repeated linear factor. We have

$$\frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

The LCD is $(3x-1)(x-1)^2$, and we multiply both sides of this identity by the LCD to get

$$\begin{aligned}
\frac{x(3-5x)}{(3x-1)(x-1)^2}\cdot(3x-1)(x-1)^2 &= \frac{A}{3x-1}\cdot(3x-1)(x-1)^2 + \frac{B}{x-1}\cdot(3x-1)(x-1)^2 \\
&\quad + \frac{C}{(x-1)^2}\cdot(3x-1)(x-1)^2 \\
x(3-5x) &= A(x-1)^2 + B(3x-1)(x-1) + C(3x-1) \\
-5x^2+3x &= A\left(x^2-2x+1\right) + B\left(3x^2-4x+1\right) + C(3x-1) \\
-5x^2+3x &= Ax^2 - 2Ax + A + 3Bx^2 - 4Bx + B + 3Cx - C \\
-5x^2+3x &= Ax^2 + 3Bx^2 - 2Ax - 4Bx + 3Cx + A + B - C \\
-5x^2+3x &= (A+3B)x^2 + (-2A-4B+3C)x + (A+B-C)
\end{aligned}$$

Since this equation is an identity, *i.e.*, it is true for all allowable values of x, we equate coefficients to get the following system of equations. There are many, many ways to solve the system. Just do it!

$$\left\{\begin{array}{rcrcrcr}
-5 &=& A &+& 3B && \\
3 &=& -2A &-& 4B &+& 3C \\
0 &=& A &+& B &-& C
\end{array}\right.$$

If we add all the equations together, we get

$$-2 = 2C$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 501, #20.

$$C = -1$$

If we substitute this value into the bottom equation, we can combine the top and bottom to solve for A and B

$$\begin{cases} -5 = A + 3B \\ -1 = A + B \end{cases}$$

If we add the top to -1 times the bottom, we get

$$-4 = 2B$$
$$B = -2$$

Finally, if we substitute our values for B and C into the bottom equation, we get

$$0 = A - 2 + 1$$
$$A = 1$$

The partial fraction decomposition is

$$\frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{1}{3x-1} + \frac{-2}{x-1} + \frac{-1}{(x-1)^2}$$

so our integral becomes

$$\begin{aligned}
\int_{x=2}^{x=3} \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x &= \int_{x=2}^{x=3} \frac{1}{3x-1} + \frac{-2}{x-1} + \frac{-1}{(x-1)^2}\,\mathrm{d}x \\
&= \int_{x=2}^{x=3} \frac{1}{3x-1}\,\mathrm{d}x + \int_{x=2}^{x=3} \frac{-2}{x-1}\,\mathrm{d}x + \int_{x=2}^{x=3} \frac{-1}{(x-1)^2}\,\mathrm{d}x \\
&= \int_{x=2}^{x=3} \frac{1}{3x-1}\,\mathrm{d}x - 2\int_{x=2}^{x=3} \frac{1}{x-1}\,\mathrm{d}x - \int_{x=2}^{x=3} \frac{1}{(x-1)^2}\,\mathrm{d}x
\end{aligned}$$

For the first integral, we'll let $u = 3x-1$, so $\mathrm{d}u = 3\mathrm{d}x$. So when $x = 2$, $u = 5$, and when $x = 3$, $u = 8$.

$$= \frac{1}{3}\int_{u=5}^{u=8} \frac{1}{u}\,\mathrm{d}u - 2\int_{x=2}^{x=3} \frac{1}{x-1}\,\mathrm{d}x - \int_{x=2}^{x=3} \frac{1}{(x-1)^2}\,\mathrm{d}x$$

The second integral is a straightforward natural logarithm. For the third integral, we'll let $w = x-1$, so $\mathrm{d}w = \mathrm{d}x$. So when $x = 2$, $w = 1$ and when $x = 3$, $w = 2$.

$$\begin{aligned}
&= \frac{1}{3}\int_{u=5}^{u=8} \frac{1}{u}\,\mathrm{d}u - 2\int_{x=2}^{x=3} \frac{1}{x-1}\,\mathrm{d}x - \int_{w=1}^{w=2} \frac{1}{w^2}\,\mathrm{d}w \\
&= \frac{1}{3}\left[\ln|u|\right]_{u=5}^{u=8} - 2\left[\ln|x-1|\right]_{x=2}^{x=3} - \left[-\frac{1}{w}\right]_{w=1}^{w=2} \\
&= \frac{1}{3}\left[\ln|8| - \ln|5|\right] - 2\left[\ln|2| - \ln|1|\right] - \left[-\frac{1}{2} - -\frac{1}{1}\right] \\
&= \frac{1}{3}\ln(8) - \frac{1}{3}\ln(5) - 2\ln(2) - \frac{1}{2} \\
&= \frac{1}{3}\ln\left(2^3\right) - \frac{1}{3}\ln(5) - 2\ln(2) - \frac{1}{2} \\
&= \ln(2) - \frac{1}{3}\ln(5) - 2\ln(2) - \frac{1}{2} \\
&= -\ln(2) - \frac{1}{3}\ln(5) - \frac{1}{2}
\end{aligned}$$

Thus,

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x = -\ln(2) - \frac{1}{3}\ln(5) - \frac{1}{2}$$