One method of slowing the growth of an insect population without the use of pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. Let P represeent the number of female insects in a population and S the number of sterile males introduced each generation. Let r be the per capita rate of production of females by females , provided their chosen mate is not sterile. Then the female population is related to time t by[1]

$$t = \int \frac{P+S}{P\left[(r-1)P - S\right]}\,\mathrm{d}P$$

Suppose an insect population with 10,000 females grows at the rate of $r = 1.1$ and 900 sterile males are added initially. Evaluate the integral to give an equation relating female population to time. (Note that the resulting equation can't be solved explicitly for P.)

Substituting the given values for r and S, we get

$$\begin{aligned} t &= \int \frac{P+900}{P\left[(1.1-1)P - 900\right]}\,\mathrm{d}P \\ &= \int \frac{P+900}{P\left[0.1P - 900\right]}\,\mathrm{d}P \end{aligned}$$

Is the integrand one of our basic indefinite integrals? No. How about a basic u-substitution? No. Integration by parts? No. Powers of trig functions? No. Does the integrand include a trig. sub. radical? No. Is the integrand a rational function? Yes!

The integrand is

$$\frac{P+900}{P(0.1P-900)} = \frac{A}{P} + \frac{B}{0.1P-900}$$

The LCD is $P(0.1P-900)$, so we multiply both sides of our identity.

$$\begin{aligned} \frac{P+900}{P(0.1P-900)} \cdot P(0.1P-900) &= \frac{A}{P} \cdot P(0.1P-900) + \frac{B}{0.1P-900} \cdot P(0.1P-900) \\ P+900 &= A(0.1P-900) + BP \\ P+900 &= 0.1AP - 900A + BP \\ P+900 &= (0.1A+B) - 900A \end{aligned}$$

We get the system of equations

$$\begin{cases} 1 = 0.1A + B \\ 900 = -900A \end{cases}$$

The second equation clearly tells us $A = -1$, and substitution into the first equation gives us $B = 1.1$.

The partial fraction decomposition is

$$\frac{P+900}{P(0.1P-900)} = \frac{-1}{P} + \frac{1.1}{0.1P-900}$$

and our integral becomes

$$t = \int \frac{P+900}{P\left[0.1P-900\right]}\,\mathrm{d}P$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 502, #67.

$$
\begin{aligned}
&= \int \frac{-1}{P} + \frac{1.1}{0.1P - 900}\, \mathrm{d}P \\
&= \int \frac{-1}{P}\, \mathrm{d}P + \int \frac{1.1}{0.1P - 900}\, \mathrm{d}P \\
&= -\int \frac{1}{P}\, \mathrm{d}P + 1.1 \cdot \int \frac{1}{0.1P - 900}\, \mathrm{d}P
\end{aligned}
$$

The first integral is a basic natural logarithm. For the second integral, let $u = 0.1P - 900$, so $\mathrm{d}u = 0.1\mathrm{d}P$.

$$
\begin{aligned}
&= -\int \frac{1}{P}\, \mathrm{d}P + 1.1 \cdot \frac{1}{0.1} \int \frac{0.1}{0.1P - 900}\, \mathrm{d}P \\
&= -\int \frac{1}{P}\, \mathrm{d}P + 1.1 \cdot 10 \int \frac{1}{u}\, \mathrm{d}u \\
&= -\ln|P| + 11\ln|u| + C \\
&= -\ln|P| + 11\ln|0.1P - 900| + C
\end{aligned}
$$

We know when $t = 0$, $P = 10{,}000$ and $S = 900$. Substituting,

$$
\begin{aligned}
0 &= -\ln(10{,}000) + 11\ln(1000 - 900) + C \\
C &= \ln(10{,}000) - 11\ln(100) \\
&= \ln\left(10^4\right) - 11\ln\left(10^2\right) \\
&= 4\ln(10) - 22\ln(10) \\
&= -18\ln(10)
\end{aligned}
$$

Thus,

$$
t = \int \frac{P + 900}{P\left[(1.1 - 1)P - 900\right]}\, \mathrm{d}P = -\ln|P| + 11\ln|0.1P - 900| - 18\ln(10)
$$