*The trouble with error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound $K$ for $\left|f^{(4)}(x)\right|$ by hand. But computer algebra systems have no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for $K$ from a machine graph. This exercise deals with approximations to the integral*[1]

$$I = \int_{-1}^{1} \sqrt{4 - x^3}\, \mathrm{d}x$$

(a) *Use a graph to get a good upper bound for $|f''(x)|$.*

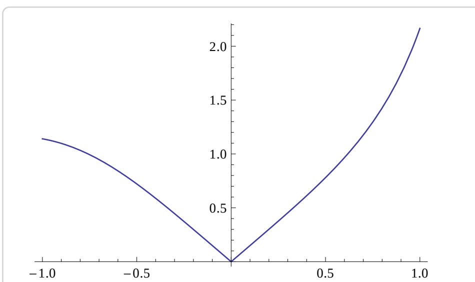
Here, $f(x) = \sqrt{4 - x^3}$ and Wolfram|Alpha (W|A) gives $f''$ as

$$\frac{d^2}{dx^2}\left(\sqrt{4 - x^3}\right) = \frac{3\,x\,(x^3 - 16)}{4\,(4 - x^3)^{3/2}}$$

D[Sqrt[4 – x^ 3], {x, 2}]

Computed by Wolfram|Alpha

so we graph the absolute value of this on the interval [−1,1].




Plot[ Abs[ D[Sqrt[4 – x^ 3], {x, 2}] ], {x, –1, 1}]

Computed by Wolfram|Alpha

From the graph, the maximum value appears to be $\approx 2.15$ when $x = 1$, but we can ask W|A for the maximum

$$-\frac{5\sqrt{3}}{4} \approx -2.16506$$

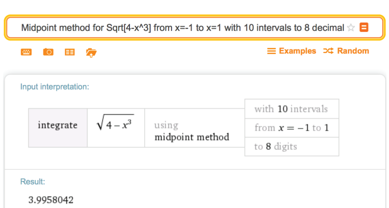
D[ Sqrt[4 – x^ 3], {x, 2} ]
when x = 1

Computed by Wolfram|Alpha

Thus a good upper bound[2] for $|f''(x)|$ is 2.1651.

(b) *Use $M_{10}$ to approximate $I$.*

From W|A,




Thus, $\int_{-1}^{1} \sqrt{4 - x^3}\, \mathrm{d}x \approx M_{10} \approx 3.9958042$.

[1]Stewart, *Calculus, Early Transcendentals*, p. 525, #24.
[2]We couldn't get W|A to give the value when asking for the absolute value, so we evaluated at $x = 1$, and then just applied the absolute value ourselves.

(c) *Use part (a) to estimate the error in part (b).*

We have

$$|E_M| \le \frac{2.1651\,(1-(-1))^3}{24 * 10^2} = 0.007217$$

(d) Use the built-in numerical integration capability of your CAS to approximate $I$.

Computation result:

$\int_{-1}^{1}\sqrt{4-x^3}\,dx$ | 10 digits | $= 3.995487677$

Thus $\int_{-1}^{1}\sqrt{4-x^3}\,\mathrm{d}x \approx 3.995487677$.

(e) *How does the actual error compare with the error estimate in part (c)?*

Actual Error $= |3.9954877 - 3.9958042| = 0.0003165$ whereas the error estimate is 0.007217. The $M_{10}$ estimate is much closer to the actual value than the error estimate indicates. In fact, the actual error is about $\frac{0.0003165}{0.007217} \approx \frac{1}{22}$ of the estimated error.

(f) *Use a graph to get a good upper bound for* $\left|f^{(4)}(x)\right|$.

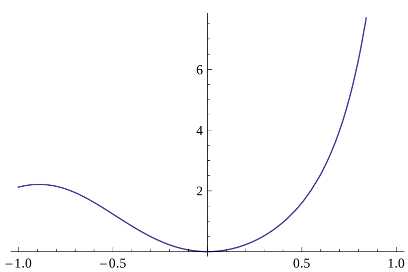
Again, $f(x) = \sqrt{4-x^3}$ and Wolfram|Alpha (W|A) gives $f^{(4)}$ as

$$\frac{d^4}{dx^4}\left(\sqrt{4-x^3}\right) = \frac{9\,x^2\,(x^6-224\,x^3-1280)}{16\,(4-x^3)^{7/2}}$$

D[Sqrt[4 − x^ 3], {x, 4}]

Computed by Wolfram|Alpha

so we graph the absolute value of this on the interval [−1,1].




Plot[ Abs[ D[ Sqrt[4 − x^ 3], {x, 4} ] ], {x, −1, 1} ]

Computed by Wolfram|Alpha

From the graph, the maximum value appears to be larger than 8 when $x = 1$, but we can ask W|A for the maximum

$-\frac{167\sqrt{3}}{16} \approx -18.0783$

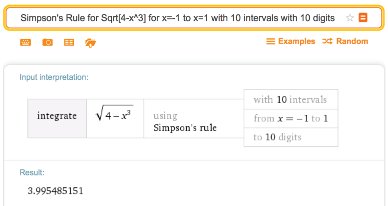
D[ Sqrt[4 − x^ 3], {x, 4} ]
when x = 1

Computed by Wolfram|Alpha

Thus a good upper bound for $\left|f^{(4)}(x)\right|$ is $|-18.0783| = 18.0783$.

(g) Use $S_{10}$ to approximate $I$.

From W|A,




Thus, $\int_{-1}^{1} \sqrt{4 - x^3}\, \mathrm{d}x \approx S_{10} \approx 3.995485151$.

(h) *Use part (f) to estimate the error in part (g).*

We have

$$|E_S| \leq \frac{18.0783\,(1 - (-1))^5}{180 * 10^4} \approx 0.0003214$$

(i) *How does the actual error compare with the error estimate in part (h)?*

Actual Error $= |3.9954877 - 3.9954852| = 0.0000025$ whereas the error estimate is $0.0003214$. The $S_{10}$ estimate is much closer to the actual value than the error estimate indicates. In fact, the actual error is about $\frac{0.0000025}{0.0003214} \approx \frac{1}{123}$ of the estimated error.

(j) How large should $n$ be to guarantee that the size of the error in using $S_n$ is less than $0.0001$?

To ensure that $|E_s| \leq 0.0001$, we use

$$\begin{aligned} |E_s| \leq \frac{18.0783\,(1-(-1))^5}{180n^4} &\leq 0.0001 \\ \frac{18.0783 * 32}{180} &\leq 0.0001n^4 \\ \frac{18.0783 * 32}{180 * 0.0001} &\leq n^4 \\ \sqrt[4]{\frac{18.0783 * 32}{180 * 0.0001}} &\leq \sqrt[4]{n^4} \\ 13.3893 &\leq n \end{aligned}$$

Since the number of intervals for Simpson's Rule must be even, we take $n = 14$ to be certain that the error in approximating the integral is less than or equal to $0.0001$.