Determine whether the integral is convergent or divergent. Evaluate the integral if it is convergent.[1]

$$\int_0^\infty \sin(\theta) \mathrm{e}^{\cos(\theta)}\, \mathrm{d}\theta$$

This is an improper integral because the interval of integration is infinite. We evaluate

$$\lim_{t\to\infty} \int_0^t \sin(\theta) \mathrm{e}^{\cos(\theta)}\, \mathrm{d}\theta$$

If we examine the integrand, $\sin(\theta)\mathrm{e}^{\cos(\theta)}$, it seems that a substitution will be necessary to find the antiderivative. Let's complete the indefinite integral first, and then we'll use that result to evaluate the improper integral.

$$\int \sin(\theta) \mathrm{e}^{\cos(\theta)}\, \mathrm{d}\theta$$

Let $u = \cos(\theta)$, so $\mathrm{d}u = -\sin(\theta)\mathrm{d}\theta$. We get

$$\begin{aligned} -1 \cdot \int -\sin(\theta)\mathrm{e}^{\cos(\theta)}\, \mathrm{d}\theta &= -\int \mathrm{e}^u\, \mathrm{d}u \\ &= -\mathrm{e}^u \\ &= -\mathrm{e}^{\cos(\theta)} \end{aligned}$$

Now we're ready to go back to the limit of our definite integral.

$$\begin{aligned} &\lim_{t\to\infty} \int_0^t \sin(\theta) \mathrm{e}^{\cos(\theta)}\, \mathrm{d}\theta \\ &\quad = \lim_{t\to\infty} \left[-\mathrm{e}^{\cos(\theta)} \right]_0^t \\ &\quad = \lim_{t\to\infty} \left[-\mathrm{e}^{\cos(t)} - -\mathrm{e}^{\cos(0)} \right] \\ &\quad = \lim_{t\to\infty} \left[-\mathrm{e}^{\cos(t)} + \mathrm{e}^1 \right] \\ &\quad = \lim_{t\to\infty} \left[-\mathrm{e}^{\cos(t)} + \mathrm{e} \right] \end{aligned}$$

As $t \to \infty$, $\cos(t)$ oscillates from -1 to 1, so $-\mathrm{e}^{\cos(t)} + \mathrm{e}$ oscillates from $-\frac{1}{\mathrm{e}} + \mathrm{e}$ to 0. Thus, the limit does not exist, and therefore the improper integral is divergent.

[1] Stewart, *Calculus, Early Transcendentals*, p. 534, #16.