Determine whether the integral is convergent or divergent. Evaluate the integral if it is convergent.[1]

$$\int_{-1}^{2} \frac{x}{(x+1)^2}\,\mathrm{d}x$$

This is an improper integral because the integrand is undefined at the lower limit of integration. We evaluate

$$\lim_{t\to -1^+} \int_{t}^{2} \frac{x}{(x+1)^2}\,\mathrm{d}x$$

Since the integrand, $\frac{x}{(x+1)^2}$ is not one of our known basic forms, let's try a substitution. (Partial fraction decomposition is also a possibility.)

Let $u = x+1$, so $\mathrm{d}u = \mathrm{d}x$ and $x = u-1$. We get the indefinite integral

$$\begin{aligned}\int \frac{x}{(x+1)^2}\,\mathrm{d}x &= \int \frac{u-1}{u^2}\,\mathrm{d}u \\ &= \int \frac{1}{u} - \frac{1}{u^2}\,\mathrm{d}u \\ &= \int \frac{1}{u} - u^{-2}\,\mathrm{d}u \\ &= \ln|u| + \frac{1}{u}\end{aligned}$$

substituting,

$$= \ln|x+1| + \frac{1}{x+1}$$

and since $x+1>0$ on the interval $(-1,2]$, we get

$$\int \frac{x}{(x+1)^2}\,\mathrm{d}x = \ln(x+1) + \frac{1}{x+1}$$

Now we're ready to go back to the limit of our definite integral.

$$\begin{aligned}&\lim_{t\to -1^+} \int_{t}^{2} \frac{x}{(x+1)^2}\,\mathrm{d}x \\ &\quad= \lim_{t\to -1^+} \left[\ln(x+1) + \frac{1}{x+1}\right]_t^2 \\ &\quad= \lim_{t\to -1^+} \left[\left(\ln(2+1) + \frac{1}{2+1}\right) - \left(\ln(t+1) + \frac{1}{t+1}\right)\right] \\ &\quad= \lim_{t\to -1^+} \left[\left(\ln(3) + \frac{1}{3}\right) - \left(\ln(t+1) + \frac{1}{t+1}\right)\right] \\ &\quad= \ln(3) + \frac{1}{3} - \lim_{t\to -1^+} \left[\ln(t+1) + \frac{1}{t+1}\right]\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 534, #30.

As $t \to -1^+$, $\ln(t+1) \to -\infty$ but $\frac{1}{t+1} \to \infty$, so our limit is indeterminate with the form $-\infty + \infty$. Let's write this in a form that allows us to apply l'Hospital's Rule.

$$= \ln(3) + \frac{1}{3} - \lim_{t \to -1^+} \frac{(t+1)\ln(t+1) + 1}{t+1} \qquad (*)$$

Unfortunately, this has the form $\frac{0 \cdot -\infty + 1}{0}$, so l'Hospital's Rule does not apply. However, we can use l'Hospital"s to compute

$$\begin{aligned}
&\lim_{t \to -1^+} (t+1)\ln(t+1) \\
&= \lim_{t \to -1^+} \frac{\ln(t+1)}{\frac{1}{t+1}} \\
&\overset{\text{(H)}}{=} \lim_{t \to -1^+} \frac{\frac{1}{t+1}}{-\frac{1}{(t+1)^2}} \\
&= \lim_{t \to -1^+} \frac{1}{t+1} \cdot -\frac{(t+1)^2}{1} \\
&= \lim_{t \to -1^+} -(t+1) \\
&= 0
\end{aligned}$$

Substituting back into (*)

$$\begin{aligned}
&= \ln(3) + \frac{1}{3} - \lim_{t \to -1^+} \frac{(t+1)\ln(t+1) + 1}{t+1} \\
&= \ln(3) + \frac{1}{3} - \frac{0+1}{0} \\
&= \ln(3) + \frac{1}{3} - \infty
\end{aligned}$$

Thus, the limit does not exist, and therefore the improper integral is divergent.