Calculus II, Section 7.8, #70
Improper Integrals

Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min*) is often well described by the equation*

$$u(t) = \frac{r}{V} C_0 \mathrm{e}^{-rt/V}$$

where r is the rate of bloodflow through the dialyzer (in mL/min*), V is the volume of the patient's blood (in* mL*), and C_0 is the amount of urea in the blood (in* mg*) at time $t = 0$. Evaluate the integral $\int_0^\infty u(t)\ \mathrm{d}t$ and interpret it.*[1]

We evaluate

$$\begin{aligned}
&\int_0^\infty \frac{r}{V} C_0 \mathrm{e}^{-rt/V}\ \mathrm{d}t \\
&\quad = \lim_{w\to\infty} \int_{t=0}^{t=w} \frac{r}{V} C_0 \mathrm{e}^{-rt/V}\ \mathrm{d}t \\
&\quad = \lim_{w\to\infty} -C_0 \int_{t=0}^{t=w} -\frac{r}{V} \mathrm{e}^{-rt/V}\ \mathrm{d}t \\
&\quad = -C_0 \lim_{w\to\infty} \int_{t=0}^{t=w} -\frac{r}{V} \mathrm{e}^{-rt/V}\ \mathrm{d}t \\
&\quad = -C_0 \lim_{w\to\infty} \left[\mathrm{e}^{-rt/V}\right]_{t=0}^{t=w} \\
&\quad = -C_0 \lim_{w\to\infty} \left[\mathrm{e}^{-r\cdot w/V} - \mathrm{e}^{-r\cdot 0/V}\right]_{t=0}^{t=w} \\
&\quad = -C_0 \lim_{w\to\infty} \left[\frac{1}{\mathrm{e}^{rw/V}} - 1\right] \\
&\quad = -C_0 \left[0 - 1\right] \\
&\quad = C_0
\end{aligned}$$

Since $u(t)$ is the rate of urea removal, $\int_0^\infty \frac{r}{V} C_0 \mathrm{e}^{-rt/V}\ \mathrm{d}t$ gives the total urea removed from the bloodflow if the dialyzer runs indefinitely. Since $\int_0^\infty \frac{r}{V} C_0 \mathrm{e}^{-rt/V}\ \mathrm{d}t = C_0$, this means all the urea will be removed from the bloodflow if the machine runs indefinitely.

[1] Stewart, *Calculus, Early Transcendentals*, p. 536, #70.