Find the exact length of the curve.[1]

$$y = \ln(\cos(x)) \qquad 0 \le x \le \frac{\pi}{3}$$

Since y is given as a function of x, we will use the arc length formula

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

We have

$$y = \ln(\cos(x))$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos(x)} \cdot -\sin(x)\\ &= -\tan(x)\end{aligned}$$

and

$$\begin{aligned}1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1 + (-\tan(x))^2\\ &= 1 + \tan^2(x)\\ &= \sec^2(x)\end{aligned}$$

Substituting into our arc length formula,

$$L = \int_{x=0}^{x=\pi/3} \sqrt{\sec^2(x)}\,\mathrm{d}x$$

Since $0 \le x \le \pi/3$, we know $\cos(x) \ge 0$, so we get

$$\begin{aligned}&= \int_{x=0}^{x=\pi/3} \sec(x)\,\mathrm{d}x\\ &= \Big[\ln\left|\sec(x) + \tan(x)\right|\Big]_{x=0}^{x=\pi/3}\\ &= \left[\ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - \ln\left|\sec(0) + \tan(0)\right|\right]\\ &= \ln\left(2 + \sqrt{3}\right) - \ln(1 + 0)\\ &= \ln\left(2 + \sqrt{3}\right)\end{aligned}$$

Thus the arc length of the curve $y = \ln(\cos(x))$ from $x = 0$ to $x = \frac{\pi}{3}$ is exactly $\ln\left(2 + \sqrt{3}\right)$.

[1] Stewart, *Calculus, Early Transcendentals*, p. 549, #14.