Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve. Compare your answer with the value of the integral produced by a calculator.[1]

$$y = \mathrm{e}^{-x^2}, \quad 0 \le x \le 2$$

Since y is given as a function of x, we will use the arc length formula

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$$

We have

$$y = \mathrm{e}^{-x^2}$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^{-x^2} \cdot -2x \\ &= -2x\mathrm{e}^{-x^2}\end{aligned}$$

and

$$\begin{aligned}1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1 + \left(-2x\mathrm{e}^{-x^2}\right)^2 \\ &= 1 + 4x^2\mathrm{e}^{-2x^2}\end{aligned}$$

Substituting into our arc length formula,

$$L = \int_{x=0}^{x=2} \sqrt{1 + 4x^2\mathrm{e}^{-2x^2}}\ \mathrm{d}x$$

For Simpson's Rule, we use

$$\int_a^b f(x)\ \mathrm{d}x \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

For our integral, $\int_{x=0}^{x=2} \sqrt{1 + 4x^2\mathrm{e}^{-2x^2}}\ \mathrm{d}x$, we have $f(x) = \sqrt{1 + 4x^2\mathrm{e}^{-2x^2}}$ and $\Delta x = \frac{2-0}{10} = \frac{1}{5}$, so

$$\begin{aligned}\int_{x=0}^{x=2} \sqrt{1 + 4x^2\mathrm{e}^{-2x^2}}\ \mathrm{d}x &\approx S_{10} \\ &= \frac{1/5}{3}\left[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8)\right. \\ &\qquad \left. + 4f(1.0) + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2.0)\right] \\ &= \frac{1}{15}\left[\sqrt{1 + 4\,(0)^2\,\mathrm{e}^{-2(0)^2}} + 4\sqrt{1 + 4\,(0.2)^2\,\mathrm{e}^{-2(0.2)^2}} + 2\sqrt{1 + 4\,(0.4)^2\,\mathrm{e}^{-2(0.4)^2}}\right. \\ &\qquad + 4\sqrt{1 + 4\,(0.6)^2\,\mathrm{e}^{-2(0.6)^2}} + 2\sqrt{1 + 4\,(0.8)^2\,\mathrm{e}^{-2(0.8)^2}} + 4\sqrt{1 + 4\,(1.0)^2\,\mathrm{e}^{-2(1.0)^2}}\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 549, #28.

$$+ 2\sqrt{1 + 4\left(1.2\right)^2 \mathrm{e}^{-2(1.2)^2}} + 4\sqrt{1 + 4\left(1.4\right)^2 \mathrm{e}^{-2(1.4)^2}} + 2\sqrt{1 + 4\left(1.6\right)^2 \mathrm{e}^{-2(1.6)^2}}$$
$$+ 4\sqrt{1 + \left(1.8\right)^2 \mathrm{e}^{-2(1.8)^2}} + \sqrt{1 + \left(2.0\right)^2 \mathrm{e}^{-2(2.0)^2}}\,\Bigg]$$
$$\approx 2.280559$$

Using `fnInt()` on our TI-84, we get

$$\int_{x=0}^{x=2} \sqrt{1 + 4x^2 \mathrm{e}^{-2x^2}}\,\mathrm{d}x \approx 2.280526$$