(a) *Set up an integral for the area of the surface obtained by rotating the curve about (i) the x-axis and (ii) the y-axis.*[1]

$$x = \ln(2y+1), \qquad 0 \le y \le 1$$

(b) *Use the numerical integration capability of a calculator to evaluate the surface areas correct to four decimal places.*

In any calculus textbook, there are a lot of formulas associated with surface area. In the eighth edition of *Calculus, Early Transcendentals* by James Stewart, there are five formulas set out for finding the surface area of a solid of revolution. If we try to intuitively understand the surface area concept, we can spare ourselves the need to memorize these formulas.

Much like the computation of the lateral area of a cylinder (circumference·length), the area of a surface of revolution is found by

circumference · arc length

where we use the formula for arc length and integrate over the length of the interval of integration.

Thus we have

$$S = \int_{x=a}^{x=b} 2\pi \cdot \text{radius} \cdot \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

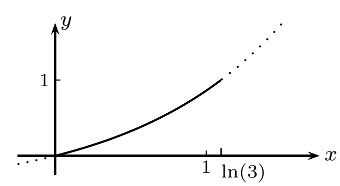
or

$$S = \int_{y=c}^{y=d} 2\pi \cdot \text{radius} \cdot \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

For this problem, we will be finding surface areas associated with revolving the graph of $x = \ln(2y+1),\ 0 \le y \le 1$ about the x and y axes. Let's do some preliminary work with the function.

The graph of $x = \ln(2y+1),\ 0 \le y \le 1$ is shown at right. A simple computation confirms when $y = 0$, $x = \ln(2\cdot 0+1) = 0$ and when $y = 1$, $x = \ln(2\cdot 1+1) = \ln(3)$. Also, if we solve for y we get

$$\begin{aligned} x &= \ln(2y+1) \\ \mathrm{e}^x &= \mathrm{e}^{\ln(2y+1)} \\ \mathrm{e}^x &= 2y+1 \\ y &= \frac{\mathrm{e}^x - 1}{2} \end{aligned}$$



Since

$$x = \ln(2y+1)$$

we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{1}{2y+1}\cdot 2 \\ \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{2}{2y+1} \end{aligned}$$

Since

$$\begin{aligned} y &= \frac{\mathrm{e}^x - 1}{2} \\ &= \frac{1}{2}\left(\mathrm{e}^x - 1\right) \end{aligned}$$

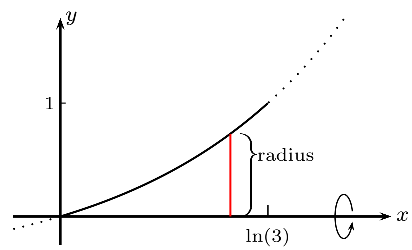
we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\mathrm{e}^x$$

With this preliminary work completed, we are ready to respond to the prompt.

[1]Stewart, *Calculus, Early Transcendentals*, p. 555, #4.

(i) The diagram at right shows the curve being revolved about the x-axis, along with a radius.



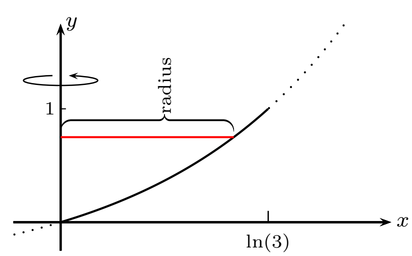
In terms of x, the radius is $y = \frac{e^x - 1}{2}$, the arc length is $\sqrt{1 + \left(\frac{1}{2}e^x\right)^2}\,dx$, and we integrate from $x = 0$ to $x = \ln(3)$. We get

$$S = \int_{x=0}^{x=\ln(3)} 2\pi \left(\frac{e^x - 1}{2}\right) \sqrt{1 + \left(\frac{1}{2}e^x\right)^2}\,dx$$
$$\approx 4.2583$$

In terms of y, the radius is y, the arc length is $\sqrt{1 + \left(\frac{2}{2y+1}\right)^2}\,dy$, and we integrate from $y = 0$ to $y = 1$. We get

$$S = \int_{y=0}^{y=1} 2\pi \cdot y \cdot \sqrt{1 + \left(\frac{2}{2y+1}\right)^2}\,dy$$
$$\approx 4.2583$$

(ii) The diagram at right shows the curve being revolved about the y-axis, along with a radius.



In terms of y, the radius is $x = \ln(2y + 1)$, the arc length is $\sqrt{1 + \left(\frac{2}{2y+1}\right)^2}\,dy$, and we integrate from $y = 0$ to $y = 1$. We get

$$S = \int_{y=0}^{y=1} 2\pi \cdot \ln(2y + 1) \cdot \sqrt{1 + \left(\frac{2}{2y+1}\right)^2}\,dy$$
$$\approx 5.6053$$

In terms of x, the radius is x, the arc length is $\sqrt{1 + \left(\frac{1}{2}e^x\right)^2}\,dx$, and we integrate from $x = 0$ to $x = \ln(3)$. We get

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