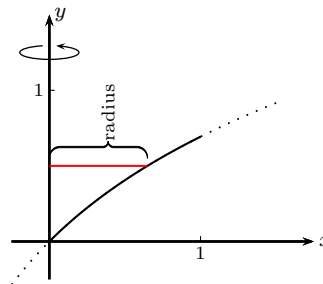


Calculus II, Section 8.2, #26
 Area of a Surface of Revolution

Use a CAS (Computer Algebra System) to find the exact area of the surface obtained by rotating the curve about the y -axis. If your CAS has trouble evaluating the integral, express the surface area as an integral in the other variable.¹

$$y = \ln(x + 1), \quad 0 \leq x \leq 1$$

The diagram at right shows the curve being revolved about the y -axis, along with a radius.



Since

$$y = \ln(x + 1)$$

we have

$$\frac{dy}{dx} = \frac{1}{x + 1}$$

So the arc length differential ds is

$$ds = \sqrt{1 + \left(\frac{1}{x + 1}\right)^2} dx$$

In terms of x , the radius is x , the arc length differential is $\sqrt{1 + \left(\frac{1}{x+1}\right)^2} dx$, and we integrate from $x = 0$ to $x = 1$. We get

$$S = \int_{x=0}^{x=1} 2\pi x \cdot \sqrt{1 + \left(\frac{1}{x + 1}\right)^2} dx$$

WolframAlpha gives

Integrate[2*Pi*x*Sqrt[1+(1/(x+1))^2], {x,0,1}]

Definite integral: $\int_0^1 2\pi x \sqrt{1 + \left(\frac{1}{x+1}\right)^2} dx = \pi(\sqrt{2} - 3 \sinh^{-1}(1) + \sinh^{-1}(2) + 2 \operatorname{csch}^{-1}(2)) \approx 3.6950$

In terms of y , the radius is $e^y - 1$, $\frac{dx}{dy} = e^y$, so the arc length differential is

$$\begin{aligned} ds &= \sqrt{1 + (e^y)^2} dy \\ &= \sqrt{1 + e^{2y}} dy \end{aligned}$$

and we integrate from $y = 0$ to $y = \ln(2)$. We get

$$S = \int_{y=0}^{y=\ln(2)} 2\pi (e^y - 1) \sqrt{1 + e^{2y}} dy$$

WolframAlpha gives

¹Stewart, *Calculus, Early Transcendentals*, p. 556, #26.

Calculus II

Area of a Surface of Revolution

Integrate[2*Pi*(E^y-1)*Sqrt[1+E^(2y)], {y,0,Log[2]}]

Assuming "Log" is the natural logarithm | Use the base 10 logarithm instead

Definite integral: More digits Step-by-step solution

$$\int_0^{\log(2)} 2\pi (e^y - 1) \sqrt{1 + e^{2y}} dy =$$
$$\pi (\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(2) - 2 \tanh^{-1}(\sqrt{2}) + 2 \tanh^{-1}(\sqrt{5}))$$
$$\approx 3.6950 + 0. \times 10^{-5} i$$

which is an interesting result because it contains a complex portion. If we ask WolframAlpha to compute the exact portion, we get

pi (sqrt(2)-sinh^(-1)(1)+sinh^(-1)(2)-2 tanh^(-1)(sqrt(2))+2 tanh^(-1)(sqrt(5)))

Assuming multiplication | Use a list instead

Input:

$$\pi (\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(2) - 2 \tanh^{-1}(\sqrt{2}) + 2 \tanh^{-1}(\sqrt{5}))$$

Decimal approximation: More digits

3.694990252151258886204591468313793483284209912118...

The approximations for the integral with respect to either x or y are both ≈ 3.6950 , so the exact area can be expressed in either way.