The centroid of a ***curve*** *can be found by a process similar to the one we used for finding the centroid of a region. If C is a curve with length L, then the centroid is $(\bar{x},\bar{y})$ where $\bar{x} = \frac{1}{L}\int x\mathrm{d}s$ and $\bar{y} = \frac{1}{L}\int y\mathrm{d}s$. Here, we assign appropriate limits of integration, and $\mathrm{d}s$ is defined as in the sections on arc length and surface area of a solid of revolution. (The centroid often doesn't lie on the curve itself. If the curve were made of wire and placed on a weightless board, the centroid would be the balance point of the board.) Find the centroid of the quarter-circle $y = \sqrt{16-x^2}$, $0 \le x \le 4$.*[1]

We could certainly use our arc length integral to find L, but since it is a quarter-circle of radius 4, we know

$$L = \frac{1}{4}\cdot 2\pi \cdot 4 = 2\pi$$

We recall

$$\mathrm{d}s = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

so

$$\begin{aligned}
\mathrm{d}s &= \sqrt{1+\left(\frac{1}{2\sqrt{16-x^2}}\cdot -2x\right)^2}\,\mathrm{d}x \\
&= \sqrt{1+\frac{x^2}{16-x^2}}\,\mathrm{d}x \\
&= \sqrt{\frac{16-x^2}{16-x^2}+\frac{x^2}{16-x^2}}\,\mathrm{d}x \\
&= \sqrt{\frac{16}{16-x^2}}\,\mathrm{d}x \\
&= \frac{4}{\sqrt{16-x^2}}\,\mathrm{d}x
\end{aligned}$$

Substituting this into the given formula for $\bar{x}$, we get

$$\bar{x} = \frac{1}{2\pi}\int_{x=0}^{x=4} x\frac{4}{\sqrt{16-x^2}}\,\mathrm{d}x$$

Let $u = 16-x^2$, so $\mathrm{d}u = -2x\,\mathrm{d}x$. Also, when $x=0$, $u=16$ and when $x=4$, $u=0$

$$\begin{aligned}
&= \frac{1}{2\pi}\cdot -2\int_{x=0}^{x=4} \frac{-2x}{\sqrt{16-x^2}}\,\mathrm{d}x \\
&= -\frac{1}{\pi}\int_{u=16}^{u=0} \frac{1}{\sqrt{u}}\,\mathrm{d}u \\
&= -\frac{1}{\pi}\int_{u=16}^{u=0} u^{-1/2}\,\mathrm{d}u \\
&= -\frac{1}{\pi}\left[2u^{1/2}\right]_{u=16}^{u=0} \\
&= -\frac{1}{\pi}\left[2\sqrt{0}-2\sqrt{16}\right] \\
&= -\frac{1}{\pi}\left[0-8\right]
\end{aligned}$$

so

$$\bar{x} = \frac{8}{\pi}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 568, #47.

For $\bar{y}$ we have

$$\begin{aligned}
\bar{y} &= \frac{1}{2\pi}\int_{x=0}^{x=4} y\frac{4}{\sqrt{16-x^2}}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{x=0}^{x=4} \sqrt{16-x^2}\cdot\frac{4}{\sqrt{16-x^2}}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{x=0}^{x=4} 4\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[4x\right]_{x=0}^{x=4} \\
&= \frac{1}{2\pi}\left[4\cdot 4 - 4\cdot 0\right]
\end{aligned}$$

so

$$= \frac{8}{\pi}$$