If income is collected continuously at a rate of $f(t)$ dollars per year, and will be invested at a constant interest rate r (compounded continuously) for a period of T years, then the ***future value*** *of the income is given by $\int_0^T f(t)\,\mathrm{e}^{r(T-t)}\,\mathrm{d}t$. Compute the future value after 6 years for income received at a rate of $f(t) = 8000\mathrm{e}^{0.04t}$ dollars per year and invested at 6.2% interest.*[1]

We are given the formula

$$\mathrm{FV} = \int_0^T f(t)\,\mathrm{e}^{r(T-t)}\,\mathrm{d}t$$

so we substitute the given values

$$\begin{aligned}
\mathrm{FV} &= \int_0^6 8000\mathrm{e}^{0.04t}\cdot\mathrm{e}^{0.062(6-t)}\,\mathrm{d}t \\
&= 8000\cdot\int_0^6 \mathrm{e}^{0.04t+0.372-0.062t}\,\mathrm{d}t \\
&= 8000\cdot\int_0^6 \mathrm{e}^{-0.022t}\cdot\mathrm{e}^{0.372}\,\mathrm{d}t \\
&= 8000\cdot\mathrm{e}^{0.372}\int_0^6 \mathrm{e}^{-0.022t}\,\mathrm{d}t \\
&= 8000\mathrm{e}^{0.372}\left[\frac{1}{-0.022}\mathrm{e}^{-0.022t}\right]_{t=0}^{t=6} \\
&= \frac{8000\mathrm{e}^{0.372}}{-0.022}\left(\mathrm{e}^{-0.022\cdot 6}-\mathrm{e}^{-0.022\cdot 0}\right) \\
&\approx \$65{,}230.48
\end{aligned}$$

The ***present value*** *of an income stream is the amount that would need to be invested now to match the future value as described above and is given by $\int_0^T f(t)\,\mathrm{e}^{-rt}\,\mathrm{d}t$. Find the present value of the income stream described above.*[2]

We are given the formula

$$\mathrm{PV} = \int_0^T f(t)\,\mathrm{e}^{-rt}\,\mathrm{d}t$$

so we substitute the given values

$$\begin{aligned}
\mathrm{PV} &= \int_0^6 8000\mathrm{e}^{0.04t}\cdot\mathrm{e}^{-0.062t}\,\mathrm{d}t \\
&= 8000\cdot\int_0^6 \mathrm{e}^{0.0.04t-0.062t}\,\mathrm{d}t \\
&= 8000\cdot\int_0^6 \mathrm{e}^{-0.022}\,\mathrm{d}t \\
&= 8000\left[\frac{1}{-0.022}\mathrm{e}^{-0.022t}\right]_{t=0}^{t=6} \\
&= \frac{8000}{-0.022}\left(\mathrm{e}^{-0.022\cdot 6}-\mathrm{e}^{-0.022\cdot 0}\right) \\
&\approx \$44{,}966.91
\end{aligned}$$

The present value tells us how much we would need to deposit now, at a rate of 6.2%, so that after the 6-year period, we would have an amount equal to the future value. This computation is frequently used if a contract is terminated before its agreed upon date. (Think Josh Hamilton and the L.A. Angels.)

[1]Stewart, *Calculus, Early Transcendentals*, p. 573, #15.
[2]Stewart, *Calculus, Early Transcendentals*, p. 573, #16.