The density function

$$f(x) = \frac{e^{3-x}}{(1+e^{3-x})^2}$$

is an example of a ***logistic distribution****.*[1]

(a) *Verify that f is a probability density function.* The probability density function $f(x)$ of a random variable X must satisfy the following two conditions:

(i) $f(x) \geq 0$ for all x.

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

The numerator of $f(x)$ is an exponential function, so it is always positive. The denominator of $f(x)$ is a square, so it is always positive. Thus $f(x) \geq 0$ for all x and condition (i) is satisfied.

We compute

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{e^{3-x}}{(1+e^{3-x})^2}\, dx \\
&\quad = \lim_{w\to-\infty} \int_w^0 \frac{e^{3-x}}{(1+e^{3-x})^2}\, dx + \lim_{t\to\infty} \int_0^t \frac{e^{3-x}}{(1+e^{3-x})^2}\, dx
\end{aligned}$$

For both of these integrals, let $u = 1 + e^{3-x}$, so $du = -1 \cdot e^{3-x}\, dx$. We get

$$\begin{aligned}
&= \lim_{w\to-\infty} -1 \cdot \int_w^0 \frac{-e^{3-x}}{(1+e^{3-x})^2}\, dx + \lim_{t\to\infty} -1 \cdot \int_0^t \frac{-e^{3-x}}{(1+e^{3-x})^2}\, dx \\
&= \lim_{w\to-\infty} -1 \cdot \left[-\left(1+e^{3-x}\right)^{-1}\right]_w^0 + \lim_{t\to\infty} -1 \cdot \left[-\left(1+e^{3-x}\right)^{-1}\right]_0^t \\
&= \lim_{w\to-\infty} \left[\frac{1}{1+e^{3-0}} - \frac{1}{1+e^{3-w}}\right] + \lim_{t\to\infty} \left[\frac{1}{1+e^{3-t}} - \frac{1}{1+e^{3-0}}\right]
\end{aligned}$$

As $w \to -\infty$, $1 + e^{3-w} \to \infty$, so $\frac{1}{1+e^{3-w}} \to 0$. Also, as $t \to \infty$, $e^{3-t} \to 0$, so $1 + e^{3-t} \to 1$, and $\frac{1}{1+e^{3-t}} \to 1$. We have

$$\begin{aligned}
&= \frac{1}{1+e^3} - 0 + 1 - \frac{1}{1+e^3} \\
&= 1
\end{aligned}$$

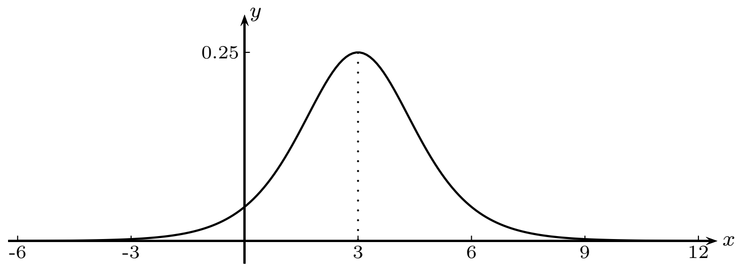
Thus, $f(x) = \dfrac{e^{3-x}}{(1+e^{3-x})^2}$ is a probability density function.

(b) Find $P(3 \leq X \leq 4)$.

$$\begin{aligned}
P(3 \leq X \leq 4) &= \int_3^4 \frac{e^{3-x}}{(1+e^{3-x})^2}\, dx \\
&= \left[\left(1+e^{3-x}\right)^{-1}\right]_3^4 \\
&= \frac{1}{1+e^{3-4}} - \frac{1}{1+e^{3-3}} \\
&= \frac{1}{1+e^{-1}} - \frac{1}{1+1} \\
&\approx 0.2311
\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 579, #4.

(c) Graph f. What does the mean appear to be? What about the median?



From the graph, it seems that the mean and the median are both 3.