(a) *Show that every member of the family of functions $y = (\ln(x) + C)/x$ is a solution to the differential equation $x^2y' + xy = 1$.*[1]

If $y = \frac{\ln(x)+C}{x}$ is a solution to $x^2y' + xy = 1$, then we should get a true statement upon substitution.

$$y = \frac{\ln(x) + C}{x}$$

so

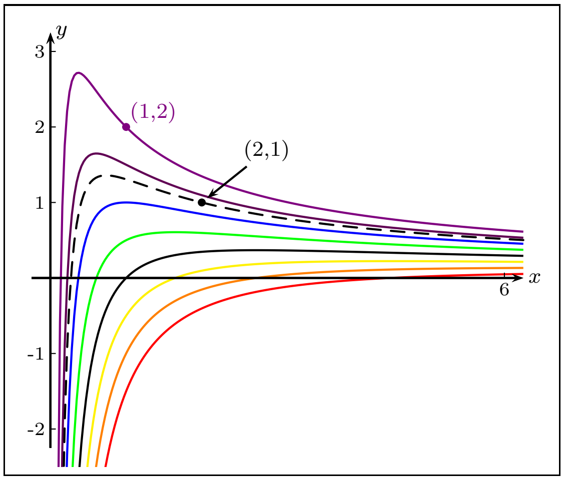
$$\begin{aligned} y' &= \frac{x \cdot \frac{1}{x} - (\ln(x) + C) \cdot 1}{x^2} \\ &= \frac{1 - \ln(x) - C}{x^2} \end{aligned}$$

Substituting into the left hand side (LHS) of $x^2y' + xy = 1$, we get

$$\begin{aligned} \text{LHS} &= x^2 \cdot \frac{1 - \ln(x) - C}{x^2} + x \cdot \frac{\ln(x) + C}{x} \\ &= 1 - \ln(x) - C + \ln(x) + C \\ &= 1 \\ &= \text{RHS} \end{aligned}$$

Thus, $y = (\ln(x) + C)/x$ is a solution to the given differential equation. Note that this is true for any value of the constant C.

(b) *Illustrate part (a) by graphing several members of the family of solutions on a common screen.*



red. . . $C = -1.5$; orange. . . $C = -1$; yellow. . . $C = -0.5$; black. . . $C = 0$; green. . . $C = 0.5$; blue. . . $C = 1$; indigo. . . $C = 1.5$; violet. . . $C = 2$; dotted. . . $C = 2 - \ln(2)$

(c) Find a solution of the differential equation that satisfies the initial condition $y(1) = 2$.

To find the particular solution that satisfies $y(1) = 2$, we will substitute $x = 1$ and $y = 2$ into the general solution $y = (\ln(x) + C)/x$ and solve for C.

$$y = \frac{\ln(x) + C}{x}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 590, #6.

Substituting $x = 1$ and $y = 2$ gives

$$2 = \frac{\ln(1) + C}{1}$$
$$2 = C$$

Thus the particular solution with $y(1) = 2$ is $y = \frac{\ln(x)+2}{x}$. On the graph from part (b), this is the violet colored solution.

(d) Find a solution of the differential equation that satisfies the initial condition $y(2) = 1$.

To find the particular solution that satisfies $y(2) = 1$, we will substitute $x = 2$ and $y = 1$ into the general solution $y = (\ln(x) + C)/x$ and solve for C.

$$y = \frac{\ln(x) + C}{x}$$

Substituting $x = 2$ and $y = 1$ gives

$$1 = \frac{\ln(2) + C}{2}$$
$$2 = \ln(2) + C$$
$$C = 2 - \ln(2)$$

Thus the particular solution with $y(1) = 2$ is $y = \frac{\ln(x)+2-\ln(2)}{x}$.